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Ring theory from symplectic geometry [1]

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Abstract

Basic results for an algebraic treatment of commutative and noncommutative Poisson algebras are described. Symplectic algebras are examined from a ring-theoretic point of view. © 1998 Elsevier Science B.V.

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0. Introduction

Our goal is to discuss the algebra involved in Poisson and symplectic geometry. There are several excellent guides to the classical commutative terrain [1, 9, 13, 17]. The desire for a noncommutative differential geometry provides an incentive to isolate the ring theory which appears explicitly and implicitly in the traditional geometry. Indeed, our own interests evolved into a research project while trying to understand the remarkable paper "Noncommutative differential geometry, quantum mechanics, and gauge theory" by Dubois-Violette. The essay we have written is a sort of primer on symplectic algebra for ring theorists, which goes far enough to clarify some of the statements in [6] and answer one of his questions. We hope that in follow-up papers we can study other symplectic themes with algebraic overtones such as Lagrangians, polarizations, group actions, and quantization.

We deal with foundations – a somewhat technical treatment of the definition of symplectic algebra – as well as properties of noncommutative algebras which stand on their own, although inspired by geometry. All of our algebras, commutative or not, are defined over a field of characteristic zero.

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In Section 1 we point out a fundamental bifurcation in the theory of Poisson algebras. While there appears to be great diversity among Poisson structures which occur for commutative algebras, we prove that if a prime Poisson algebra is not commutative, then the Poisson bracket must be the commutator bracket up to appropriate scalar.

Section 2 is devoted to a general definition of symplectic algebra, modeled after $C^\infty(M)$ for a symplectic manifold M. We allow substantial latitude in what it means to be a "differential two-form" which induces a duality between tangent and cotangent vectors. This flexibility allows us to identify the most general notion of symplectic algebra which distills the algebraic essence of many symplectic propositions.

It turns out that the general version of symplectic algebra is broad enough to encompass algebras of differential operators on commutative affine domains. As a consequence, we prove in Section 3 that if $\mathscr{D}(B)$ is such an algebra, then the commutator of two associative algebra derivations of $\mathscr{D}(B)$ is inner. (This generalizes Dixmier's theorem that the derivations of the Weyl algebra are inner.) Moreover, if B is regular then the Lie algebra of inner derivations is precisely the derived algebra of $Der(\mathscr{D}(B))$; this is a noncommutative instance of a theorem of Calabi [2]. Next, we analyze the idea of a formally infinite local differential alternating two-form on the Weyl algebra, which was introduced by Dubois-Violette. We show that although his definition provides a context for some wonderful formulas, all alternating forms on the Weyl algebra have infinite expansions as formal differentials. Finally, we prove that if a simple algebra has a symplectic structure supported by a finite differential expression, then the algebra satisfies a polynomial identity.

In Section 4, we concentrate on more classical commutative Poisson algebras. Given some extra smoothness assumptions, we show that the Poisson bracket is compatible with a symplectic structure when the module of all algebra derivations is generated by Poisson-inner derivations. Cotangent algebras, the crucial examples of regular symplectic algebras, are carefully examined.

The two-form ω which supports a symplectic algebra is required to be closed with respect to some differential. Say that ω is a symplectic potential when ω is exact. We answer a question of Dubois-Violette by observing that the local differential two-form which supports the Weyl algebra as a symplectic algebra is not a symplectic potential. Our analysis suggests that not-commutative algebras whose commutator bracket is compatible with a symplectic structure have the remarkable property that all of their Lie algebra derivations are associative algebra derivations. Other connections with the Lie structure of a symplectic algebra are explored.

1. Poisson algebras

We shall restrict our attention to algebras over a field k of characteristic zero. All associative algebras will have a multiplicative identity element.

Definition. A *Poisson algebra* A is an associative algebra which is, at the same time, a Lie algebra under a *Poisson bracket* $\{*,*\}$. Operations are related by requiring that the bracket be an associative algebra derivation in each argument (e.g, the Leibniz rule $\{ab,c\} = a\{b,c\} + \{a,c\}b$ holds for all $a,b,c \in A$.)

Commutative Poisson algebras appear in a variety of geometric and algebraic contexts. We single out a few prototypes. Verifying details such as the Jacobi identity are left to the reader.

Example 1. Assume $\mathcal{G}$ is a finite-dimensional Lie algebra over k. The symmetric algebra $k[\mathcal{G}]$ is isomorphic to the polynomial ring in $dim_k\mathcal{G}$ indeterminates. The Lie bracket on $\mathcal{G}$ can be extended uniquely via the Leibniz rule so that $k[\mathcal{G}]$ becomes a Poisson algebra.

Example 2. Assume that V is a finite-dimensional vector space over k and that $\mathfrak{B}$ is an alternating bilinear form on V. We denote the symmetric algebra on V by $k[\mathfrak{B}]$ and extend $\mathfrak{B}$ to a Poisson bracket on all of $k[\mathfrak{B}]$. For instance, if V is a two-dimensional vector space with basis elements S and T where $\mathfrak{B}(S,T) = 1$ then $k[\mathfrak{B}] = k[S,T]$ with $\{S,T\} = 1$. This algebra is, in some sense, the canonical example of a Poisson algebra; it is the coordinatization of the “symplectic plane”.

Example 3. Suppose R is a noncommutative filtered algebra, i.e., $R_0 \subseteq R_1 \subseteq R_2 \subset \cdots$ with $\bigcup R_j = R$ and $R_iR_j \subseteq R_{i+j}$. If the associated graded algebra is commutative, then it is a Poisson algebra. The bracket on grR is obtained as follows. If a lies in the i^{th} homogeneous component of grR and b lies in the j^{th} component, then a pulls back to some $\alpha \in R_i$ and b pulls back to some $\beta \in R_j$. Define $\{a,b\}$ to be the image of the commutator $[\alpha,\beta] = \alpha\beta - \beta\alpha$ in the $(i+j-1)$th component of grR.

If $\mathcal{G}$ is a finite-dimensional Lie algebra and $U = U(\mathcal{G})$ is its universal enveloping algebra, then U is filtered by U_n, which is the span of all products of m members of $\mathcal{G}$ for $m \leq n$. As a Poisson algebra, grU is isomorphic to $k[\mathcal{G}]$ [5, Ch. 2].

Consider the Weyl algebra $\mathbf{A}_1$ which is generated as an algebra by p and q with the relation $[q,p] = 1$. (This may be familiar to some readers as a simple image of the Heisenberg algebra.) It is well known that $\mathbf{A}_1$ has a basis consisting of all p^mq^n for $m,n \leq 0$. If $\mathbf{A}_1$ is filtered by the degree in q, then the associated graded algebra is isomorphic as a Poisson algebra to the example $k[S,T]$ discussed above.

Example 4. Any associative algebra is a Poisson algebra under the commutator bracket.

In reading Dubois-Violette’s survey article [6], we were struck by the lack of any fundamental not-commutative Poisson algebra whose bracket is not the commutator. Much to our surprise, we discovered a bifurcation: all prime Poisson algebras which are not commutative fall under Example 4, “up to scalar multiple”. To be more precise, recall that a ring A is prime provided that the product of nonzero ideals is nonzero;

the extended centroid $\mathscr{Z}^+(A)$ of a prime ring A is the center of the Martindale ring of quotients of A. (See [8, 1.3] for the construction.) For example, if A is a simple k-algebra, then $\mathscr{Z}^+(A)$ is the center of A, which we will denote $\mathscr{Z}(A)$. When A is a prime Goldie ring, then $\mathscr{Z}^+(A) = \mathscr{Z}$ (Fract (A)). In general, we know that $\mathscr{Z}^+$ is a field [8, 1.3.1].

Our characterization of not-commutative Poisson algebras rests on an "exchange formula".

Lemma 1.1. *If A is any Poisson algebra, then $[a,c]\{b,d\} = \{a,c\}[b,d]$ for all $a,b,c,d \in A$.*

Proof. On the one hand,

$$\begin{aligned}\{ab,cd\} &= a\{b,cd\} + \{a,cd\}b \\ &= ac\{b,d\} + a\{b,c\}d + c\{a,d\}b + \{a,c\}db.\end{aligned}$$

On the other hand,

$$\begin{aligned}\{ab,cd\} &= c\{ab,d\} + \{ab,c\}d \\ &= ca\{b,d\} + c\{a,d\}b + a\{b,c\}d + \{a,c\}bd. \qquad \square\end{aligned}$$

Theorem 1.2. *If A is a prime not-commutative Poisson algebra then there is a $\lambda \in \mathscr{Z}^+(A)$ such that $\{c,d\} = \lambda[c,d]$ for all $c,d \in A$.*

Proof. We begin by extending the exchange formula. For $a,b,c,d,x \in A$

$$[a,b]\{xc,d\} = \{a,b\}[xc,d].$$

Hence

$$[a,b]x\{c,d\} + [a,b]\{x,d\}c = \{a,b\}x[c,d] + \{a,b\}[x,d]c.$$

However, the second terms on each side of the equality are themselves equal by the exchange formula. Therefore,

$$(*) \qquad [a,b]x\{c,d\} = \{a,b\}x[c,d] \quad \text{for all } a,b,c,d,x \in A.$$

In particular,

$$[a,b]x\{a,b\} = \{a,b\}x[a,b] \quad \text{for all } a,b \in A.$$

Since A is not commutative, we can find a and b with $[a,b] \neq 0$. Under this choice, Corollary 1.3.2 of [8] applies: there exists a $\lambda \in \mathscr{Z}^+$ such that $\{a,b\} = \lambda[a,b]$.

To handle arbitrary elements c and d of A we revisit $(*)$. Substituting,

$$[a,b]x\{c,d\} = \lambda[a,b]x[c,d] \quad \text{for all } x \in A.$$

Since $\mathscr{Z}^+$ centralizes $\mathscr{Z}^+ \cdot A$, we may replace x with an arbitrary member of $\mathscr{Z}^+ \cdot A$. Consequently,

$$[a,b]y(\{c,d\} - \lambda[c,d]) = 0 \text{ for all } y \in \mathscr{Z}^+ \cdot A.$$

The theorem follows because $\mathscr{Z}^+ \cdot A$ is a prime ring. □

(A version of the theorem is valid for semiprime algebras since there is still a notion of extended centroid. Under the added assumption that some commutator $[a,b]$ is not a zero divisor, the same conclusion holds.)

As an application, consider the Weyl algebra $\mathbf{A}_1$. It is simple and its center is k. Hence, any Poisson bracket on $\mathbf{A}_1$ is associated with a scalar $\lambda \in k$ such that $\{a,b\} = \lambda[a,b]$ for all $a,b \in \mathbf{A}_1$. The algebra of $n \times n$ matrices over k has the same property.

One may ask if it is truly necessary to worry about scalars from $\mathscr{Z}^+$. Look at the three-dimensional nilpotent Lie algebra spanned by x, y and z with z central and $[x,y] = z$. Let U be its universal enveloping algebra. Then $[U,U]U = zU$. (This is a consequence of the general observation that $[U(\mathscr{G}), U(\mathscr{G})]U(\mathscr{G}) = U_+([\mathscr{G},\mathscr{G}])U(\mathscr{G})$ where U_+ denotes the augmentation ideal.) Thus, it makes sense to define a Poisson bracket on U by $\{a,b\} = z^{-1}[a,b]$.

There is a way to resolve the apparent problem of a lack of noncommutative Poisson algebras: find a more appropriate definition of general Poisson algebra. This is done in [18]. We will not follow this direction in the present paper.

2. Symplectic algebras

Historically, the notion of Poisson algebra was invented to abstract fundamental formal properties of $C^\infty(M)$ for a symplectic manifold M. Such a manifold is characterized by having a bijective pairing, at each point, of a tangent vector with a cotangent vector, which is induced by a closed differential 2-form. Recalling that vector fields on M act as derivations of $C^\infty(M)$, the rough idea is to use derivations to "contract" the 2-form so that a rich supply of 1-forms is obtained. (Informally, the contraction of 2-forms is a ratcheting up of the classical identification of cotangent vectors df as functionals on tangent vectors, namely $\langle \partial/\partial x, df \rangle = \partial f/\partial x$. One may think of the derivation $\partial/\partial x$ as contracting a 1-form df to a 0-form $\partial f/\partial x$.) Several authors [1, 6, 13] have come up with the general notion of symplectic algebra to capture the geometry of a symplectic manifold in the ring $C^\infty(M)$. There are competing definitions which are not all equivalent. Rather than take sides, we will give a definition which includes all proposals. The price we pay is a somewhat technical description of appropriate candidates for an "algebra of differentials".

The reader put off by abstractions should skip down to the examples.

We always consider algebras over a field k of characteristic zero. If A is a k-algebra, then $Der_k(A)$ denotes the Lie algebra of all k-linear associative algebra derivations from

A to itself. To those accustomed to studying commutative algebras, it is worth warning that $Der_k(A)$ is not generally an A-module; rather, it is a $\mathscr{Z}(A)$-module.

Our "differentials" will reside in a differential non-negatively graded k-algebra Δ equipped with a degree $+1$ differential d. The candidate for "contraction" map will be a DGA-derivation of Δ. By this we mean a k-multilinear map f, homogeneous of degree -1, such that for homogeneous elements $\alpha, \beta \in \Delta$ we have

$$f(\alpha\beta) = f(\alpha)\beta + (-1)^{deg\alpha}\alpha f(\beta).$$

As an immediate consequence, DGA-derivations of Δ are left Δ^0-module maps.

Definition. Let A be a k-algebra. A differential graded k-algebra (Δ, d) with $\Delta^0 = A$ is a *source of differentials* for A provided that for each $X \in Der_k(A)$ there is a DGA-derivation i_X of Δ such that for all $X, Y \in Der_k(A)$

(i) $i_X(da) = X(a)$ for all $a \in A$;
(ii) the map sending X to i_X is left $\mathscr{Z}(A)$-linear (i-linearity);
(iii) $i_X i_Y + i_Y i_X = 0$ (transposition); and
(iv) $[i_X d + d i_X, i_Y] = i_{[X,Y]}$ (bracket).

In the differential geometry literature i_X is called the "interior product" as well as the "contraction" while $i_X d + d i_X$ is the "Lie derivative" associated to X. For the reader concerned about the formulas (iii) and (iv), it is worth noting that in many examples Δ is generated by Δ^0 and Δ^1; in such cases the DGA-derivation property of i_X uniquely determines the contraction, making these formulas calculations.

We are indebted to J. Stasheff for pointing out that our notion of a source of differentials is essentially H. Cartan's definition of a DGA algebra over $\mathscr{Z}(A)$ which is operated on by a Lie algebra, namely $Der_k(A)$ ([3]). Cartan replaces (iii) with the equivalent assumption that each i_Z has square zero and adds the illuminating observation that the map which sends X to $i_X d + d i_X$ is a Lie algebra morphism. (While this is an axiom in [3], it readily follows from (iii) and (iv).)

There are four fundamental examples of sources of differentials to keep in mind.

Example 1. *Alt*

Define $Alt^n(A)$ to be the vector space of $\mathscr{Z}(A)$-multilinear alternating maps from $(Der_k(A))^n$ to A. (Strictly speaking, we should write $Alt^n_{\mathscr{Z}(A)}(Der_k(A), A)$ but we hope no ambiguity will arise with the abbreviated notation.) If $\omega \in Alt^r(A)$ and $\omega' \in Alt^s(A)$, then the product $\omega\omega'$ lies in $Alt^{r+s}(A)$ with

$$(\omega\omega')(X_1, \ldots, X_{r+s})$$
$$= \frac{1}{r!}\frac{1}{s!} \sum_{\sigma \in Sym(r+s)} (-1)^\sigma \omega(X_{\sigma(1)}, \ldots, X_{\sigma(r)})\omega'(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}).$$

The differential d is defined by

$$d\omega(X_0,\ldots,X_n)=\sum_t(-1)^t X_t(\omega(X_0,\ldots,\hat{X}_t,\ldots,X_n))$$
$$+\sum_{r<s}(-1)^{r+s}\omega([X_r,X_s],\, X_0,\ldots,\hat{X}_r,\ldots,X_j,\ldots,\hat{X}_s,\ldots,X_n)$$

where ^ means "omit". For example, if $a \in A = Alt^0(A)$ then

$$da(X_0)=X_0(a).$$

Note that we have described the classical Chevalley–Eilenberg cochain complex for Lie algebra cohomology.

The contraction i_Y is given by

$$(i_Y(\omega))(X_1,\ldots,X_n)=\omega(Y,X_1,\ldots,X_n).$$

We shall see that $Alt(A)$ is, in some sense, the most general source of differentials.

Example 2. *DAlt*

$DAlt(A)$ is the differential graded subalgebra of $Alt(A)$ generated by A. To make this more concrete, recall that if $a,b \in A$ then $(da)b = d(ab) - a(db)$. Thus, every element of $DAlt^n(A)$ is a finite sum of elements with the form $a_0da_1da_2\cdots da_n$ for $a_0,\ldots,a_n \in A$. In this case one can use the formula describing a DGA-derivation to arrive at the (usual) explicit formula

$$i_X(a_0da_1\cdots da_n)=\sum_{t=1}^{n}(-1)^{t-1}a_0\; da_1\; da_2\cdots d\breve{a}_t\cdots da_n$$

where $d\breve{a}_t$ denotes the replacement of this symbol with $X(a_t)$.

Example 3. Ω

It may have occurred to the reader that $DAlt(A)$ is a generalization of the de Rham algebra $\Omega(A)$ (the exterior algebra on Kähler differentials) for commutative algebras. This is not quite the case, which the reader should be aware of when comparing with [6]. Assume that A is commutative and let $(-)^*$ denote the dual $Hom_A(-,A)$. Then $DAlt^1(A)$ is nothing but $(Der_k(A))^*$. However, $\Omega^1(A)$ is the module of Kähler differentials and $Der_k(A) \simeq (\Omega^1(A))^*$. (Amusingly enough, the isomorphism is given by sending X to i_X.) Thus, $DAlt(A)$ can be identified with $\Omega(A)$ only when $\Omega^1(A)$ is reflexive, the usual problem of double duals. Nonetheless, the formulas for d and i_X in $\Omega(A)$ are formally the same as those in $DAlt(A)$. Moreover, the distinction we are making disappears when A is regular.

Example 4. $U\Omega$

Connes [4] has brought into prominence a universal noncommutative version of Ω. In this construction, $U\Omega^0$ is obtained by adjoining a new identity element to A. The failure to have the bottom component coincide with A does not fatally affect the potential of $U\Omega$ as a source of differentials. In this paper, however, we do not study the universal construction and, so, leave this difficulty for another time.

Definition. Let A be a k-algebra and assume Δ is a source of differentials for A. Then A is *Δ-symplectic* provided there is a $\omega \in \Delta^2$ such that $d\omega = 0$ and dA lies in the image of the linear map $Der_k(A) \to \Delta^1$ which sends X to $i_X(\omega)$. In this case we say that ω *supports* the symplectic structure on A.

In [13], Loose requires that the map sending X to $i_X(\omega)$ be injective. This requirement is redundant.

Theorem 2.1 (Nondegeneracy). *Let A be a Δ-symplectic algebra supported by ω. The map $Der_k(A) \to \Delta^1$ which sends X to $i_X(\omega)$ is injective.*

Proof. For $a \in A$ we may choose a derivation $ham(a)$ of A such that $i_{ham(a)}(\omega) = d(-a)$ in Δ^1. If $i_X(\omega) = 0$ then

$$0 = i_{ham(a)} i_X(\omega) = -i_X i_{ham(a)}(\omega) = -i_X d(-a) = X(a).$$

Since $X(a) = 0$ for all $a \in A$, we conclude that $X = 0$. $\square$

If it happens that A is supported by $\omega \in Alt^2(A)$, then Theorem 2.1 says that the map which sends X to $\omega(X, -)$ is injective : the form ω is nondegenerate.

Some comments are in order about $ham(a)$. First of all, "ham" is an abbreviation for "hamiltonian vector field". Second, $ham(a)$ is uniquely determined by a in that there is only one derivation Z such that $i_Z(\omega) = d(-a)$. (This is immediate from the theorem.) Since d and the map sending X to i_X are k-linear, uniqueness forces $ham : A \to Der_k(A)$ to be k-linear.

We now connect an arbitrary source of differentials Δ with $Alt(A)$. For $n > 0$, $\omega \in \Delta^n$ and $X_1, \dots, X_n \in Der_k(A)$, define

$$\widetilde{\omega}(X_1, \dots, X_n) = i_{X_n} \cdots i_{X_2} i_{X_1}(\omega).$$

Then $\widetilde{\omega}$ takes values in A because i_X reduces degree by one. It is $\mathscr{X}(A)$-multilinear by i-linearity and alternating by the transposition property. Thus $\widetilde{\omega} \in Alt^n(A)$. Since the contraction map is a left A-module homomorphism, the assignment of ω to $\widetilde{\omega}$ is a left A-module map. The proof of the next theorem is technically unpleasant and best left to a second reading.

Theorem 2.2. *Let Δ be a source of differentials for $A = \Delta^0$.*

(a) *If $a \in A$, then $(da)^\sim$ agrees with da in $Alt^1(A)$.*

(b) *If $\theta \in \Delta^1$, then $d\widetilde{\theta} = (d\theta)^\sim$.*

(c) *If $\omega \in \Delta^2$, then $d\widetilde{\omega} = (d\omega)^\sim$.*

Proof. Let $X, Y, Z \in Der_k(A)$.

(a) $(da)^{\sim}(X) = i_X(da) = X(a) = (da)(X)$ where the first da lies in Δ^1 and the second lies in $Alt^1(A)$.

(b) $$\begin{aligned} d\widetilde{\theta}(X,Y) &= X\widetilde{\theta}(Y) - Y\widetilde{\theta}(X) - \widetilde{\theta}([X,Y]) \\ &= i_X d i_Y(\theta) - i_Y d i_X(\theta) - i_{[X,Y]}(\theta) \\ &= i_Y i_X d(\theta) - d i_X i_Y(\theta), \quad \text{using the bracket property.} \end{aligned}$$

But $i_X i_Y(\Delta^1) = 0$ and $i_Y i_X(d\theta) = (d\theta)^{\sim}(X,Y)$.

(c) This calculation is tricky. It depends on the identity $i_T i_{[U,V]} + i_{[U,V]} i_T = 0$ for Δ. To temporarily simplify notation, we will use upper and lower case letters so that $i_B = b$. Expand the left side of the identity using transposition and bracket for sources of differentials, to obtain

$$(*) \quad dtuv + (tduv + udvt + vdtu) + (tudv + uvdt + vtdu) + tuvd = 0.$$

We wish to calculate

$$\begin{aligned} d\widetilde{\omega}(X,Y,Z) &= X\widetilde{\omega}(Y,Z) + Y\widetilde{\omega}(Z,X) + Z\widetilde{\omega}(X,Y) \\ &\quad -\widetilde{\omega}([X,Y],Z) - \widetilde{\omega}([Y,Z],X) - \widetilde{\omega}([Z,X],Y) \\ &= (xdzy + ydxz + zdyx - z[x,y] - x[y,z] - y[z,x])(\omega). \end{aligned}$$

Now $[x,y] = dxy + xdy - ydx - yxd$, so

$$\begin{aligned} d\widetilde{\omega}(X,Y,Z) = (&2xdzy + 2ydxz + 2zdyx + 2zydx + 2xzdy + 2yxdz \\ &+zyxd + xzyd + yxzd)(\omega). \end{aligned}$$

Use the transposition property for contractions to write

$$zyxd + xzyd + yxzd = 3zyxd.$$

Finally, apply $(*)$ along with the observation that

$$(zyx)(\omega) = 0 \quad \text{for } \omega \in \Delta^2.$$

We substitute

$$d\widetilde{\omega}(X,Y,Z) = -2zyxd(\omega) + 3zyxd(\omega) = zyxd(\omega). \qquad \square$$

It is possible to iterate formula $(*)$ to obtain the general assertion that $d\widetilde{\omega} = (d\omega)^{\sim}$ for all ω.

Corollary 2.3. *Suppose $a_j, b_j, c_j \in A$. If $\phi = \sum a_j db_j$ in Δ^1, then $\widetilde{\phi} = \sum a_j db_j$ in $Alt^1(A)$. If $\mu = \sum a_j db_j dc_j$ in Δ^2, then $\widetilde{\mu} = \sum a_j db_j dc_j$ in $Alt^2(A)$.*

Proof. For the first assertion,

$$\begin{aligned} \widetilde{\phi} &= \sum a_j \widetilde{db_j} \quad \text{by } A\text{-linearity} \\ &= \sum a_j db_j \quad \text{by } (a) \text{ of the theorem.} \end{aligned}$$

As to the second, $\mu = \sum a_j d(b_j dc_j)$, so

$$\begin{aligned}\widetilde{\mu} &= \sum a_j (d(b_j dc_j))^{\sim} && \text{by } A\text{-linearity}\\ &= \sum a_j d(b_j dc_j)^{\sim} && \text{by } (b) \text{ of the theorem}\\ &= \sum a_j d(b_j dc_j) && \text{by the first half of the proof}\\ &= \sum a_j db_j dc_j. \quad \square\end{aligned}$$

Corollary 2.4. *If A is a Δ-symplectic algebra supported by ω, then A has an Alt^2-symplectic structure supported by $\widetilde{\omega}$. Moreover, the functions $ham : A \to Der_k(A)$ defined relative to ω and $\widetilde{\omega}$ are identical.*

Proof. If $a \in A$ then $i_{ham(-a)}(\omega) = da$. Thus $i_X \circ i_{ham(-a)}(\omega) = X(a)$. We can interpret this last equality in $Alt(A)$:

$$\widetilde{\omega}(ham(-a), X) = X(a), \quad \text{i.e., } i_{ham(-a)}(\widetilde{\omega}) = da \text{ in } Alt(A).$$

This proves da is some contraction of $\widetilde{\omega}$ and establishes the nonambiguity of *ham*.
Finally, $d\omega = 0$ implies $d\widetilde{\omega} = 0$, with the help of Theorem 2.2c. $\square$

The second corollary says that Alt-symplectic structures are the most general possible. From now on, the unprefixed term "symplectic" will mean Alt-symplectic.

At long last, we produce a Poisson algebra. If A is a Δ-symplectic k-algebra supported by ω then A is a Poisson algebra under the bracket

$$\begin{aligned}\{a,b\} &= i_{ham(b)} i_{ham(a)}(\omega) \quad \text{for } a,b \in A.\\ &= \widetilde{\omega}(ham(a), ham(b)).\end{aligned}$$

It is obvious that the bracket is alternating. The bracket is bilinear because *ham* is linear. By applying the basic equality of Corollary 2.4,

$$\widetilde{\omega}(X, ham(u)) = X(u)$$

and the fact $ham(a)$ is a derivation, we derive

$$\{a, bc\} = b\{a,c\} + \{a,b\}c.$$

Jumping the gun a bit, notice that "*ham*" coincides with "*ad*" if we forget the associative algebra structure on A. (The reason for the new jargon is that for not-commutative Poisson algebras "*ad*" is ambiguous – it may refer to $\{*,*\}$ or the commutator $[*,*]$.) It is well known that the Jacobi identity is equivalent to *ad* being a Lie homomorphism. Hence, the Poisson bracket satisfies the Jacobi identity provided $[ham(a), ham(b)] = ham\{a,b\}$ for all $a,b \in A$. This is a consequence of the cocycle requirement $d\widetilde{\omega} = 0$:

$$0 = d\widetilde{\omega}(X, ham(a), ham(b)) = X\{a,b\} - ham(a)X(b) + ham(b)X(a)$$
$$-[X, ham(a)](b) - \widetilde{\omega}([ham(a), ham(b)], X) - [ham(b), X](a).$$

Thus,

$$0 = X\{a,b\} - \{a, X(b)\} + \{b, X(a)\} - (X\{a,b\} - \{a, X(b)\})$$
$$-\widetilde{\omega}([ham(a), ham(b)], X) - (\{b, X(a)\} - X\{b,a\}).$$

Simplifying, we obtain

$$\widetilde{\omega}(X, [ham(a), ham(b)]) = X\{a,b\}.$$

But $X\{a,b\} = \widetilde{\omega}(X, ham\{a,b\})$. We are finished by the nondegeneracy of $\widetilde{\omega}$.

Definition. Let A be any Poisson algebra. For each $a \in A$ define $ham(a) : A \to A$ by $ham(a)(b) = \{a,b\}$. Let $Ham(A)$ denote $\{ham(a) | a \in A\}$. A *Poisson derivation* of A is an associative algebra derivation which is, at the same time, a Lie algebra derivation for the bracket $\{*,*\}$. Denote the set of all Poisson derivations by $PDer(A)$.

We return to the "*ad*/*ham* identification". If x is an element of a Lie algebra $\mathscr{G}$, then $ad(x)$ is a Lie algebra derivation of $\mathscr{G}$ and $ad\ \mathscr{G}$ is a Lie ideal in the Lie algebra of all Lie derivations of $\mathscr{G}$. (Indeed, $[\delta, ad(x)] = ad(\delta(x))$.) Notice that $PDer(A)$ is a Lie algebra because it is the intersection of the Lie algebras of associative and Lie derivations. As a consequence, $Ham(A)$ is a Lie ideal of $PDer(A)$.

As might be expected, $PDer(A)/Ham(A)$ is the first cohomology group for some cohomology theory, at least when A is commutative [9]. We will not pursue this further. Instead, we record a lemma attributed to R. Palais in the commutative case [2].

Proposition 2.5. *If A is a symplectic algebra then*

$$[PDer(A), PDer(A)] \subseteq Ham(A).$$

Proof. Assume that the symplectic structure is supported by $\omega \in Alt^2(A)$. We show that if $X, Y \in PDer(A)$ then

$$[X, Y] = ham\ \omega(X, Y).$$

First notice that, since $X \in PDer(A)$, we have

$$[ham(a), X](u) = \{a, X(u)\} - X\{a,u\} = -\{X(a), u\} = -(hamX(a))(u)$$

for all $u \in A$. We apply the cocycle property in the case of the three derivations $ham(a), X$, and Y and we use the observation above,

$$ham(a)\omega(X, Y) + XY(a) - YX(a) + \omega(hamX(a), Y)$$
$$- [X, Y](a) - \omega(hamY(a), X) = 0,$$

i.e.

$$ham(a)\omega(X,Y) + [X,Y](a) = 0.$$

But

$$\begin{aligned} ham(a)\omega(X,Y) &= \{a, \omega(X,Y)\} \\ &= -\{\omega(X,Y), a\} \\ &= -ham(\omega(X,Y))(a). \end{aligned}$$

Thus $[X,Y](a) = ham(\omega(X,Y))(a)$. Since a is arbitrary, we have the desired formula. □

Calabi [2] has proved that when $A = C^{\infty}(M)$ for a symplectic manifold M there is equality

$$[PDer(A), PDer(A)] = Ham(A).$$

We do not know to what extent this is true for algebraic examples. (But see Corollary 3.6.)

Definition. Let A be a Poisson algebra. The *Poisson center* of A, denoted $\mathscr{P}\mathscr{Z}(A)$, is

$$\{a \in A | \{a,b\} = 0 \text{ for all } b \in A\}.$$

The Poisson center is both a Lie ideal and an associative subalgebra of A. When the Poisson bracket is the commutator bracket, we have $\mathscr{P}\mathscr{Z}(A) = \mathscr{Z}(A)$.

Proposition 2.6. *If A is a symplectic algebra, then every derivation of A vanishes on the Poisson center.*

Proof. Suppose the symplectic structure is supported by ω. For any $x \in Der_k(A)$ and $a \in A$,

$$\omega(X, ham(a)) = X(a).$$

If $a \in \mathscr{P}\mathscr{Z}(A)$ then $ham(a) = 0$. The result follows. □

Propositions 2.5 and 2.6 may be the key ingredients of an internal ring-theoretic description of those symplectic algebras whose Poisson bracket coincides with the commutator bracket. In what follows, we adopt the traditional terminology for $Ham(A)$ when using the commutator; it is the Lie algebra of inner derivations $IDer_k(A)$. If the symplectic structure on A induces the commutator bracket, every associative algebra derivation is automatically a Poisson derivation. Thus, in this case,

$$[Der_k(A), Der_k(A)] \subseteq IDer_k(A)$$

by Proposition 2.5. Also, all derivations of A vanish on $\mathscr{Z}(A)$ by Proposition 2.6.

Suppose, as a special case, that every algebra derivation of A is actually inner. Then A is symplectic under the well-defined "commutator" form

$$\omega(ad(a), ad(b)) = [a, b] \quad \text{for all } a, b \in A.$$

Obviously, the Poisson bracket is the commutator bracket. We can do somewhat better.

Theorem 2.7. *Let B be a k-algebra such that*
(a) *all algebra derivations of B are zero on $\mathscr{Z}(B)$; and*
(b) *$Der_k(B) = IDer_k(B) + \mathscr{A}$ where $\mathscr{A}$ is an abelian Lie subalgebra of $Der_k(B)$.*
Then B has a symplectic structure whose Poisson bracket is the commutator bracket.

Notice that hypothesis (b) implies that $[X, Y] \in IDer_k(B)$ for all $X, Y \in Der_k(B)$.

Proof. We use "ad" for the commutator bracket. The argument depends on the elementary observation that $ad(a) = ad(b)$ for $a, b \in B$ if and only if $a - b \in \mathscr{Z}(B)$. Thus there is an alternating $\mathscr{Z}(B)$-bilinear form $\langle *, * \rangle : IDer_k(B) \times IDer_k(B) \to B$ given by $\langle ad(c), ad(d) \rangle = [c, d]$. Hypothesis (a) tells us there is a $\mathscr{Z}(B)$-bilinear form $\langle *, * \rangle : Der_k(B) \times IDer_k(B) \to B$ given by $\langle X, ad(a) \rangle = X(a)$.

According to hypothesis (b), a vector space complement $\mathscr{L} \subseteq \mathscr{A}$ to $IDer_k(B)$ in $Der_k(B)$ gives rise to a Lie algebra direct sum

$$Der_k(B) = IDer_k(B) \oplus \mathscr{L}$$

with $\mathscr{L}$ an abelian Lie algebra. If $X \in Der_k(B)$, write $X = X_0 + X_1$ following this direct sum. For $X, Y \in Der_k(B)$ set

$$\omega(X, Y) = \langle X_0, Y_0 \rangle + \langle X_1, Y_0 \rangle - \langle Y_1, X_0 \rangle.$$

Clearly, ω is alternating and $\mathscr{Z}(B)$-bilinear. Moreover, if $X_0 = ad(a)$ then

$$\begin{aligned} \omega(X, ad(b)) &= \langle ad(a), ad(b) \rangle + \langle X_1, ad(b) \rangle \\ &= [a, b] + X_1(b) \\ &= (ad(a) + X_1)(b) \\ &= X(b). \end{aligned}$$

In order to check the cocycle condition on ω, we use a trick borrowed from the theory of rings of differential operators. Imbed B in $End_k(B)$ by associating each $b \in B$ to left multiplication by b. For each $x \in Der_k(B)$ choose $b_X \in B$ such that $ad(b_X) = X_0$. Define $h_X \in End_k(B)$ by

$$h_X = b_X + X_1.$$

Using hypothesis (b) we see that

$$\begin{aligned}[h_X,h_Y] &= [b_X,b_Y]+[X_1,b_Y]+[b_X,Y_1]+[X_1,Y_1]\\ &= [b_X,b_Y]+[X_1,b_Y]-[Y_1,b_X]\\ &= \omega(X,Y).\end{aligned}$$

To prove $d\omega = 0$ notice that, for $v \in B$,

$$\begin{aligned}X(v) &= (ad(b_X))(v)+X_1(v)\\ &= [b_X,v]+[X_1,v]\\ &= [h_X,v].\end{aligned}$$

Hence, $X(\omega(Y,Z)) = [h_X,[h_Y,h_Z]]$. Thus, the sum of the first three terms of the cocycle expression vanishes by the Jacobi identity. A second calculation shows that $[X,Y] = ad\omega(X,Y)$. Hence,

$$-\omega([X,Y]Z) = \omega(Z, ad\omega(X,Y)) = Z(\omega(X,Y)) = [h_Z,[h_X,h_Y]].$$

The last three terms of the cocycle expression are also the terms of the Jacobi identity. □

3. Not-commutative algebras

At best, we would like to determine all algebras possessing a symplectic structure. At worst, we would like to have large classes of algebras which are (or which are not) symplectic. These two problems depend on the particular source of differentials we choose. In the exposition below, we proceed from general to special sources.

Throughout this section we examine prime algebras which are not commutative. By virtue of Theorem 1.2, there is no real loss of generality in assuming that if there is a symplectic structure, the Poisson bracket is the commutator.

We first show that if A is a commutative k-affine domain then the ring $\mathcal{D}(A)$ of differential operators on A is symplectic. We defer to [14] for many details. Recall that $\mathcal{D}(A)$ consists of k-endomorphisms of A; in regarding A as a left $\mathcal{D}(A)$-module we write $f * a$ for the evaluation of f at a. It is frequently more tractable to calculate in the "little" ring of differential operators, denoted $\mathbf{D}(A)$, which is the subalgebra of $\mathcal{D}(A)$ generated by A and $Der_k(A)$. For instance, $\mathbf{D}(A)Der_k(A) * 1 = 0$. As a consequence, we have the following identity for $u \in \mathbf{D}(A)Der_k(A)$ and $a \in A$:

$(\diamondsuit)\qquad [u,a] * 1 = u * a.$

(Indeed, $[u,a] * 1 = u * (a * 1) - a * (u * 1) = u * a - 0$.) A second consequence is that $\mathbf{D}(A) = A \oplus \mathbf{D}(A)Der_k(A)$. Putting these two observations together, we see that A is a maximal commutative subalgebra of $\mathbf{D}(A)$.

Definition. Let A be a commutative k-algebra. Then

$$\mathscr{I}(A) = \{\zeta \in Der_k(\mathbf{D}(A)) | \zeta(A) = 0\}.$$

Lemma 3.1. *$\mathscr{I}(A)$ is an abelian Lie subalgebra of $Der_k(\mathbf{D}(A))$.*

Proof. Let $\zeta \in \mathscr{I}(A)$. Then ζ may be applied to $Der_k(A)$ considered as a subset of $\mathbf{D}(A)$. Since $[Der_k(A), A] \subseteq A$ we have

$$\begin{aligned}[\zeta(Der_k(A)), A] &= [\zeta(Der_k(A)), A] + [Der_k(A), \zeta(A)] \\ &= \zeta([Der_k(A), A]) \\ &\subseteq \zeta(A) = 0.\end{aligned}$$

Thus, $\zeta(Der_k(A))$ centralizes A in $\mathbf{D}(A)$. By the remark above,

$$\zeta(Der_k(A)) \subseteq A.$$

Suppose $\zeta_1, \zeta_2 \in \mathscr{I}(A)$. To test that $[\zeta_1, \zeta_2] = 0$, we need only test this derivation on algebra generators of $\mathbf{D}(A)$, i.e., on $A \cup Der_k(A)$. The commutator obviously vanishes on A. Furthermore, if $X \in Der_k(A)$, then $\zeta_2(X) \in A$, so $(\zeta_1 \circ \zeta_2)(X) = 0$. We conclude that $[\zeta_1, \zeta_2](X) = 0$. $\square$

Theorem 3.2. *Let L be a finitely generated field extension of k. Then*

$$Der_k(\mathbf{D}(L)) = IDer_k(\mathbf{D}(L)) + \mathscr{I}(L).$$

Proof. We may write $\mathbf{D}(L) = L[q_1, \ldots, q_n]$ where $p_1, \ldots, p_n$ is a transcendence base for L over k and $q_j = \partial/\partial p_j$ (see [14, 15.2.5]). Let $\alpha \in Der_k(\mathbf{D}(L))$. Since $[p_i, p_j] = 0$,

$$0 = \alpha([p_i, p_j]) = [\alpha(p_i), p_j] - [\alpha(p_j), p_i].$$

That is,

$$[\alpha(p_i), p_j] = [\alpha(p_j), p_i].$$

It is easy to check that if $lq_1^{\varepsilon_1} \cdots q_n^{\varepsilon_n} \in \mathbf{D}(L)$ with $l \in L$, then

$$[lq_1^{\varepsilon_1} \cdots q_n^{\varepsilon_n}, p_j] = \frac{\partial}{\partial q_j}(lq_i^{\varepsilon_1} \cdots q_n^{\varepsilon_n}),$$

where the differentiation takes place as if l were a constant and the q_i were commuting variables. Thus,

$$\frac{\partial}{\partial q_j}(\alpha(p_i)) = \frac{\partial}{\partial q_i}(\alpha(p_j)) \quad \text{for all } i, j.$$

It is a standard result from elementary calculus – the existence of potentials for gradients – that there must be a "polynomial" $h \in L[q_1, \ldots, q_n]$ such that

$$\frac{\partial}{\partial q_i}(h) = \alpha(p_i) \quad \text{for all } i.$$

(To be pedestrian, it is easy to formally integrate polynomials in the q_j. For example, when $n = 3$ set $h_1(q_1, q_2, q_3) = \int \alpha(p_1) dq_1$ holding q_2 and q_3 fixed, set $h_2(q_2, q_3) = \int(\alpha(p_2) - \partial h_1/\partial q_2) dq_2$ holding q_3 fixed, and set $h_3(q_3) = \int(\alpha(p_3) - (\partial h_1/\partial q_3) - \partial h_2/\partial q_3) dq_3$. Let $h = h_1 + h_2 + h_3$.)

We have $[h, p_j] = \alpha(p_j)$ for $j = 1, \ldots, n$. By the unique extension of derivations to fields of quotients and finite field extensions (in characteristic zero), we conclude that α and $ad(h)$ agree when restricted to L. In other words, $\alpha - ad(h) \in \mathscr{J}(L)$. □

Now assume that L is the field of fractions of A. In order to descend from $\mathscr{D}(L)$ to $\mathscr{D}(A)$, we need to apply a result of [16] to the effect that $\mathscr{D}(L)$ is a classical left ring of quotients of $\mathscr{D}(A)$. (We sketch a proof by induction that if $f \in \mathscr{D}_m(L)$, the mth level of the filtration, then there is a nonzero $s \in A$ with $sf \in \mathscr{D}(A)$. Let $a_1, \ldots, a_t$ be k-algebra generators for A. By induction there is an $s \in A \backslash \{0\}$ such that $s[f, a_i] \in \mathscr{D}(A)$ for all i and $sf * 1 \in A$. By the Leibniz product formula, $s[f, A] \subset \mathscr{D}(A)$ whence $[sf, A] \subset \mathscr{D}(A)$. If $a \in A$ then $(sf) * a = [sf, a] * 1 + a(sf) * 1$. Since $sf * A \subset A$ we have $sf \in \mathscr{D}(A)$).

Theorem 3.3. *Let A be a commutative k-affine domain with field of fractions L. Then*

$$Der_k(\mathscr{D}(A)) = IDer_k(\mathscr{D}(A)) + \mathscr{J}(A).$$

Proof. By the remarks preceding the theorem, derivations of $\mathscr{D}(A)$ extend uniquely to derivations of $\mathscr{D}(L)$. But $\mathscr{D}(L) = \mathbf{D}(L)$ [14, 15.5.6]. Thus, if $\alpha \in Der_k(\mathscr{D}(A))$, we may extend it and write $\alpha = ad(h) + T$ in $Der_k(\mathbf{D}(L))$ with $h \in \mathbf{D}(L)$ and $T \in \mathscr{J}(L)$, according to the previous theorem. Since $\mathbf{D}(L) = L \oplus \mathbf{D}(L)Der_k(L)$ and $adL \subseteq \mathscr{J}(L)$, we may assume $h \in \mathbf{D}(L)Der_k(L)$ (i.e., absorb the adL contribution into T).

Now $\alpha(A) \subseteq \mathscr{D}(A)$ and $T(A) = 0$ because $A \subseteq L$. Hence, $[h, A] \subseteq \mathscr{D}(A)$. By identity ($\diamondsuit$) in L, we have $h * A \subseteq A$. But

$$\mathscr{D}(A) = \{f \in \mathscr{D}(L) | f * A \subseteq A\}$$

[14, 15.5.5]. Thus, $h \in \mathscr{D}(A)$. Also, $(\alpha - ad(h))(\mathscr{D}(A)) \subseteq \mathscr{D}(A)$ implies $T(\mathscr{D}(A)) \subseteq \mathscr{D}(A)$. We conclude that $\alpha = ad(h) + T|\mathscr{D}(A)$ inside $Der_k(\mathscr{D}(A))$. □

Corollary 3.4. *If A is a commutative k-affine domain then $\mathscr{D}(A)$ has a symplectic structure compatible with the commutator bracket.*

Proof. We shall apply Theorem 2.7 with $B = \mathscr{D}(A)$. Its hypothesis (b) is the content of the previous theorem. To verify hypothesis (a) we must identify the center of $\mathscr{D}(A)$. It is enough to know that the center is a finite field extension of k because derivations which vanish on k also vanish on any finite extension.

There is no loss of generality in replacing A with its field of fractions L. Write $\mathbf{D}(L) = L[q_1, \ldots, q_n]$ as in Theorem 3.2. Since the center of $\mathbf{D}(L)$ centralizes L, it lies in L. Thus, $a \in \mathscr{Z}(\mathbf{D}(L))$ satisfies a minimal polynomial over $k(p_1, \ldots, p_n)$:

$$a^m + f_{m-1}a^{m-1} + \cdots + f_1 a + f_0 = 0 \quad \text{with } f_j \in k(p_1, \ldots, p_n).$$

If any f_t lies outside k, we can find a q_s such that

$$[q_s, f_t] = \frac{\partial}{\partial p_s}(f_t) \neq 0.$$

Therefore,

$$\frac{\partial}{\partial p_s}(f_{m-1})a^{m-1} + \cdots + \frac{\partial}{\partial p_s}(f_1)a + \frac{\partial}{\partial p_s}(f_0) = 0$$

with at least one nonzero coefficient. This contradiction to minimality ensures that a is algebraic over k. $\square$

We can get more information if we assume that A is a regular affine domain. There are many equivalent formulations of regularity; the one that will be most useful for us is that $\Omega^1(A)$ is a finitely generated projective A-module [11, 7.4].

Recall that if P is a finitely generated projective A-module, then P has a projective basis $(p_1, f_1), (p_2, f_2), \ldots, (p_m, f_m)$ where $p_j \in P^*$. By this we mean that $x = \sum f_j(x)p_j$ for all $x \in P$. If L is the field of fractions of A, then the rank of P is the dimension of $L \otimes_A P$ as a vector space over L. But the rank can also be calculated intrinsically. If $char\, A = 0$, the rank is the trace of the identity endomorphism of P, i.e., $\sum f_j(p_j)$.

Now suppose that $P = \Omega^1(A)$, so that $(\Omega^1(A))^* \simeq Der_k(A)$. We will identify $i_X \in (\Omega^1(A))^*$ with $X \in Der_k(A)$ and choose a projective basis $(f_1, X_1), (f_2, X_2), \ldots, (f_m, X_m)$ for $\Omega^1(A)$, where $X_j \in Der_k(A)$. It follows that $(X_1, \widetilde{f}_1), (X_2, \widetilde{f}_2), \ldots, (X_m, \widetilde{f}_m)$ is a projective basis for $Der_k(A)$ with

$$\widetilde{f}_j(Y) = i_Y(f_j).$$

(If $f = \sum u_i dv_i$, then $\widetilde{f}(Y) = \sum u_i Y(v_i)$.)

We can do somewhat better. If $(f_1, X_1), (f_2, X_2), \ldots, (f_m, X_m)$ is a projective basis for $\Omega^1(A)$, then we may assume that $f_j = b_j dc_j$ for some b and c in A, then $(dc_1, b_1X_1), (dc_2, b_2X_2), \ldots (dc_m, b_mX_m)$ is also a projective basis for $\Omega^1(A)$. Set $Y_j = b_jX_j$. Then $(Y_1, \widetilde{da_1}), (Y_2, \widetilde{da_2}), \ldots, (Y_m, \widetilde{da_m})$ is a projective basis for $Der_k(A)$. In other words, $Z = \sum Z(a_j)Y_j$ for all $Z \in Der_k(A)$. With a slight abuse of notation, we shall write this projective basis as $(Y_1, a_1), (Y_2, a_2), \ldots, (Y_m, a_m)$.

We revisit Calabi's Theorem.

Theorem 3.5. *Let A be a commutative regular k-affine domain. Then*

$$[\mathscr{D}(A), \mathscr{D}(A)] = \mathscr{D}(A).$$

Proof. We will freely use the theorem that $\mathscr{D}(A) = \mathbf{D}(A)$ for regular A [14, 15.5.6].

Recall that $\mathbf{D} = \mathbf{D}(A)$ has a natural filtration

$$\mathbf{D}_0 \subset \mathbf{D}_1 \subset \mathbf{D}_2 \subset \cdots,$$

where $\mathbf{D}_m = (Der_k(A))^m + \mathbf{D}_{m-1}$. Let $(Y_1, a_1), (Y_2, a_2), \ldots, (Y_n, a_n)$ be a projective basis for $Der_k(A)$ with $a_j \in A$. We claim that

$$(*) \quad \text{if } f \in \mathbf{D}_m \quad \text{then } \sum_s [f, a_s] Y_s \equiv mf \pmod{\mathbf{D}_{m-1}}.$$

Induct on m. If $m = 0$ the assertion amounts to the fact that A is commutative. For $m = 1$, it is the fundamental property of a projective basis. To prove that it holds for arbitrary m, we may assume that $f = f_1 f_2$ where $f_1 \in Der_k(A)$ and $f_2 \in (Der_k(A))^{m-1}$:

$$\begin{aligned}
\sum_s [f, a_s] Y_s &= \sum [f_1 f_2, a_s] Y_s \\
&= \sum f_1 [f_2, a_s] Y_s + \sum [f_1, a_s] f_2 Y_s \\
&\equiv f_1 \sum [f_2, a_s] Y_s + \sum [f_1, a_s] Y_s f_2 \pmod{\mathbf{D}_{m-1}}. \\
&\equiv (m-1) f_1 f_2 + f_1 f_2 \pmod{\mathbf{D}_{m-1}} \\
&\equiv mf \pmod{\mathbf{D}_{m-1}}.
\end{aligned}$$

We complete the proof of the theorem with a second induction, showing that if $g \in \mathbf{D}_m$, then $g \in [\mathbf{D}, \mathbf{D}]$. Begin the induction at $m = -1$, with the convention that $\mathbf{D}_{-1} = 0$. Assume the truth of the assertion for $m - 1$. If $g \in \mathbf{D}_m$, then $(*)$ and the rank formula yield

$$\begin{aligned}
\sum_s [g Y_s, a_s] &= \sum [g, a_s] Y_s + \sum g [Y_s, a_s] \\
&\equiv mg + rank Der_k(A) \cdot g \pmod{\mathbf{D}_{m-1}}.
\end{aligned}$$

But $[g Y_s, a_s] \in [\mathbf{D}, \mathbf{D}]$. Thus, induction implies

$$(m + rank Der_k(A)) \cdot g \in [\mathbf{D}, \mathbf{D}].$$

The result follows because the rank is greater than zero. □

Corollary 3.6. *If A is a regular k-affine domain, then*

$$[Der_k(\mathscr{D}(A)), Der_k(\mathscr{D}(A))] = IDer_k(\mathscr{D}(A)).$$

Proof. According to Corollary 3.4 and Proposition 2.5, we need only check that $IDer_k(\mathscr{D}(A)) \subseteq [Der_k(\mathscr{D}(A)), Der_k(\mathscr{D}(A))]$. Any inner derivation has the form $ad\ g$ for some g in $\mathscr{D}(A)$. By the theorem,

$$ad\ g \in ad[\mathscr{D}(A), \mathscr{D}(A)] = [IDer_k(\mathscr{D}(A)), IDer_k(\mathscr{D}(A))]. \qquad \square$$

Given Corollary 3.4, we might become greedy and ask, for example, if the Weyl algebra $\mathbf{A}_1$ is supported by some ω given by a "formula with differentials". In fact, there is a remarkable formula which comes from the Moyal quantization for the symplectic plane. Recall that $\mathbf{A}_1 = k[p,q]$ subject to the relation $[q,p]=1$. If $u,v \in \mathbf{A}_1$ then ([6])

$$[u,v] = \sum \frac{(-1)^n}{n!}([\underbrace{u,p,p,\dots,p}_{n\ p\text{'s}}][\underbrace{v,q,q,\dots,q}_{n\ q\text{'s}}] - [\underbrace{v,p,\dots,p}_{n\ p\text{'s}}][\underbrace{u,q,\dots,q}_{n\ q\text{'s}}]),$$

where $[a,b,b,b,\dots,b] = [\cdots[[[a,b],b],b]\cdots b]$. Since derivations of $\mathbf{A}_1$ are inner ([Dix]), the associated commutator form is

$$\omega = \sum_n \frac{(-1)^n}{n!}[\underbrace{dp,p,p,\dots,p}_{n\ p\text{'s}}][\underbrace{dq,q,\dots,q}_{n\ q\text{'s}}].$$

(Note that in counting the p's (or q's) we include the one to the right of the d.)

Both of these expressions require some explanation since they are formally infinite series. They make sense because all but finitely many terms vanish when evaluated on a fixed element of $\mathbf{A}_1$. Based on this observation, Dubois-Violette introduces a new source of differential forms.

Definition. Let A be a k-algebra. $LDAlt^m(A)$, the local differential alternating forms, consists of those $\omega \in Alt^m(A)$ such that for each finite dimensional subspace V of $Der_k(A)$ there is a $\omega_V \in DAlt^m(A)$ such that $\omega|V^m = \omega_V$.

The choice of ω given in the formula above resides in $LDAlt^2(\mathbf{A}_1)$. It turns out that, although the formula is quite wonderful, the source of differentials can be thought of in more elementary terms. We shall argue that some formula of the type we have exhibited must exist because $LDAlt(\mathbf{A}_1) = Alt(\mathbf{A}_1)$.

Our analysis rests on a single calculation in $\mathbf{A}_1$. For each $(m,n) \in \mathbb{N}\times\mathbb{N}$ with $(m,n) \neq (0,0)$ let $X_{(m,n)}$ be the derivation $ad(p^m q^n)$. The collection of all $X_{(m,n)}$ constitutes a k-basis for $Der_k(\mathbf{A}_1)$. Define $D(m,n) \in Alt^1(\mathbf{A}_1)$ by

$$D(m,n) = \begin{cases} [\underbrace{dp,p,p,\dots,p}_{n\ p\text{'s}},\underbrace{q,\dots,q}_{m\ q\text{'s}}] & \text{if } n > 0,\\ [\underbrace{dq,\dots,q}_{m\ q\text{'s}}] & \text{if } n = 0. \end{cases}$$

The formula below can be proved directly by induction.

$$(*) D(m,n)(X_{(u,v)})$$
$$= \begin{cases} (-1)^m u(u-1)\cdots(u-m+1)v(v-1)\cdots(v-n+1)rp^{u-m}q^{v-n} \\ \quad \text{if } m \le u \text{ and } n \le v, \\ 0 \text{ otherwise.} \end{cases}$$

To exploit the formula, we will need a lengthy digression which is a particularly simple triangular instance of Moebius inversion.

Proposition 3.7. *Let V be a vector space over k and let $\{v_\alpha | \alpha \in \mathscr{A}\}$ be a basis for V well-ordered by $(\mathscr{A}, \leq)$. Assume that A is a k-algebra and that for each $\alpha \in \mathscr{A}$, there is an $f^\alpha \in Hom_k(V,A)$ such that*

(i) given $\beta \in \mathscr{A}$, there are only finitely many γ such that $f^\gamma(v_\beta) \neq 0$;

(ii) $f^\alpha(v_\beta) = 0$ for $\alpha > \beta$;

(iii) $f^\alpha(v_\alpha)$ is always a unit in A.

Then every member of $Hom_k(V,A)$ can be written uniquely as a formally infinite sum $\sum_{\alpha \in \mathscr{A}} a_\alpha f^\alpha$ with $a_\alpha \in A$.

Proof. First observe that $\sum_\alpha a_\alpha f^\alpha$ makes sense by condition (i). Consider the issue of uniqueness next. We must show that if $\sum_\alpha a_\alpha f^\alpha = 0$, then each $a_\alpha = 0$. If not, choose β minimal such that $a_\beta \neq 0$. By (ii), $(\sum_\alpha a_\alpha f^\alpha)(v_\beta) = a_\beta f^\beta(v_\beta)$. Condition (iii) tells us that $a_\beta f^\beta(v_\beta) \neq 0$, a contradiction.

As to existence, suppose that $g \in Hom_k(V,A)$. We define coefficients a_α inductively. If ε is the smallest element of $\mathscr{A}$ set $a_\varepsilon = g(v_\varepsilon)(f^\varepsilon(v_\varepsilon))^{-1}$. If σ is the smallest element of $\mathscr{A}$ for which a_σ has not yet been defined, set

$$a_\sigma = \left(g(v_\sigma) - \sum_{\tau < \sigma} a_\tau f^\tau(v_\sigma) \right) (f^\sigma(v_\sigma))^{-1}.$$

We have designed $h = \sum_{\alpha \in \mathscr{A}} a_\alpha f^\alpha$ so that

$$h(v_\sigma) = \sum_{\tau \leq \sigma} a_\tau f^\tau(v_\sigma) = g(v_\sigma) \quad \text{for all } \sigma \in \mathscr{A}. \qquad \square$$

We claim that if the hypotheses of Proposition 3.7 are assumed for V and $Hom_k(V,A)$, then we can extend the conclusion of the proposition to $Alt(V^n, A)$, the space of alternating n-forms from V to A. For $\bar{\alpha}, \bar{\beta} \in \mathscr{A}^n$ we write $\bar{\beta} = (\beta(1), \ldots, \beta(n))$ and $v_{\bar{\beta}} = (v_{\beta(1)}, v_{\beta(2)}, \ldots, v_{\beta(n)}) \in V^n$. Define $f^{\bar{\alpha}} \in Hom_k(V^n, A)$ by

$$f^{\bar{\alpha}}(v_{\bar{\beta}}) = \sum_{\pi \in Sym(n)} (-1)^\pi f^{\alpha(1)}(v_{\beta\pi(1)}) f^{\alpha(2)}(v_{\beta\pi(2)}) \cdots f^{\alpha(n)}(v_{\beta\pi(n)}).$$

Then $f^{\bar{\alpha}}$ is a k-multilinear alternating function. Consequently, $f^{\bar{\alpha}}$ is determined by its action on the subset of the basis consisting of all $v_{\bar{\beta}}$ such that $\beta(1) > \beta(2) > \cdots > \beta(n)$. Set $\wedge^n \mathscr{A}$ to be the corresponding subset of $\mathscr{A}^n$ consisting of those n-tuples whose coordinates are in strictly decreasing order; set W to be the span of $\{v_{\bar{\beta}} | \bar{\beta} \in \wedge^n \mathscr{A}\}$. It is easy to verify that $\wedge^n \mathscr{A}$ is a well-ordered set under the left-to-right lexicographic order. We examine each of the conditions of the proposition for W and $Hom_k(W,A)$.

Condition (i) obviously holds. Suppose $\bar{\alpha} > \bar{\beta}$; we may assume that the first $t-1$ coordinates of $\bar{\alpha}$ and $\bar{\beta}$ agree but $\alpha(t) > \beta(t)$. If $\pi \in Sym(n)$ and $\pi(t) \geq t$, then $\beta\pi(t) \leq \beta(t)$, so $\beta\pi(t) < \alpha(t)$. Hence $f^{\alpha(t)}(v_{\beta\pi(t)}) = 0$. If $\pi(t) < t$, then there is an s with $1 \leq s \leq t-1$ and $\pi(s) \geq t$. Now $\alpha(s) = \beta(s) > \beta(t) \leq \beta(\pi(s))$, so $f^{\alpha(s)}(v_{\beta\pi(s)}) = 0$. Since each term of $f^{\bar{\alpha}}(v_{\bar{\beta}})$ vanishes, we have $f^{\bar{\alpha}}(v_{\bar{\beta}}) = 0$: condition (ii) is verified. Finally, if $\pi \neq 1$, then there must be an s with $\pi(s) > s$, whence $\alpha\pi(s) < \alpha(s)$. Thus

$f^{\alpha(s)}(v_{\alpha\pi(s)}) = 0$. It follows that $f^{\bar{\alpha}}(v_{\bar{\alpha}}) = f^{\alpha(1)}(v_{\alpha(1)}) \cdots f^{\alpha(n)}(v_{\alpha(n)})$, a product of units in A.

We summarize our observation.

Corollary 3.8. *Under the hypotheses of the proposition, every member of $Alt(V^n, A)$ can be written uniquely as a formally infinite sum*

$$\sum_{\bar{\alpha} \in \Lambda^n \mathscr{A}} a_{\bar{\alpha}} f^{\bar{\alpha}}; \quad a_{\bar{\alpha}} \in A.$$

As promised, we can now dispense with the distinction between local and global *Alt*.

Theorem 3.9. *$LD\,Alt(\mathbf{A}_1) = Alt(\mathbf{A}_1)$.*

Proof. Set $\mathscr{A} = \{(m,n) \in \mathbb{N}^2 | (m,n) \neq (0,0)\}$ and give it the lexicographic order. (One can also order by first comparing the sum of the coordinates and then using the lexicographic order.) The three conditions of Proposition 3.7 are immediate from formula $(*)$ for $D(m,n)$. $\square$

Using the notation we have developed, the alternating bilinear form which induces the commutator bracket is

$$\sum_{n>0} \frac{(-1)^n}{n!} D(0,n) D(n,0).$$

Finally, we investigate prime not-commutative algebras which are $D\,Alt$-symplectic. Dubois-Violette has observed that if k is a field of characteristic zero, then the algebra $M_n(k)$ of $n \times n$ matrices over k is a $D\,Alt$-symplectic algebra whose Poisson bracket coincides with the commutator bracket [6]. This can be derived abstractly using the semisimplicity of the Lie algebra $\mathscr{G}l(k)$. However, if E_{ij} are the $n \times n$ matric units, then one can readily verify the following identity:

$$[A,B] = -\frac{1}{n} \sum_{i,j} ([A, E_{ij}][B, E_{ji}] - [B, E_{ij}][A, E_{ji}]).$$

Since all derivations of $M_n(k)$ are inner, we see that the commutator form is given by

$$\omega = -\frac{1}{n} \sum_{i,j} (dE_{ij})(dE_{ji}).$$

We suspect that, in spirit, this example accounts for all prime not-commutative $D\,Alt$-symplectic algebras.

Theorem 3.10. *Let A be a not-commutative $D\,Alt$-symplectic algebra whose Poisson bracket is nonzero. If A is either a prime algebra with finite uniform dimension or a* *simple algebra, then every derivation of A vanishes on the center of A and A satisfies a polynomial identity.*

Proof. According to the main theorem of Section 1, there is a nonzero element λ in the extended centroid $\mathscr{Z}^+$ of A such that

$$\{a,b\} = \lambda[a,b] \quad \text{for all } a,b \in A.$$

Thus, if $a \in \mathscr{Z}(A)$ we have $a \in \mathscr{P}\mathscr{Z}(A)$. We conclude from Proposition 2.6 that derivations of A vanish on $\mathscr{Z}(A)$.

Let $\sum a_i db_i dc_i$ represent the alternating form which supports the symplectic structure. Then

$$\sum a_i[r,b_i][s,c_i] - \sum a_i[s,b_i][r,c_i] = \lambda[r,s]$$

for all $r,s \in A$. By linearity, this equality holds for all $r,s \in \mathscr{Z}^+ \cdot A$. Formally, it appears that $\mathscr{Z}^+ \cdot A$ satisfies a generalized polynomial identity of degree 2 [8]. However, we must check that

$$\sum a_i[T,b_i][S,c_i] - \sum a_i[S,b_i][T,c_i] - \lambda[T,S]$$

is not identically zero. If this expression is zero in the free algebra over $\mathscr{Z}^+ \cdot A$, then the sum of terms which have T to the left of S must be zero:

$$\sum a_i[T,b_i][S,c_i] - \lambda TS \equiv 0.$$

Specializing S to 1 yields $-\lambda T = 0$. However, $\lambda \neq 0$ because the Poisson bracket is not always zero.

We now invoke Martindale's Theorem [8, 1.3.2]. It states that there is an idempotent $e \in \mathscr{Z}^+ \cdot A$ such that $\mathscr{Z}^+ \cdot A$ is a primitive ring with minimal right ideal $e(\mathscr{Z}^+ \cdot A)$ and that $e(\mathscr{Z}^+ \cdot A)e$ is a finite-dimensional division algebra over $\mathscr{Z}^+$.

If A is a simple algebra then $\mathscr{Z}^+ \cdot A = A$ and the existence of a minimal right ideal forces A to be simple artinian [8, 1.2.2]. Thus, there is a division algebra D finite dimensional over $\mathscr{Z}(A)$ such that $A \simeq M_n(D)$. It follows that A satisfies a standard identity.

Suppose A is prime with finite uniform dimension. Let I be a nonzero right ideal of $\mathscr{Z}^+ \cdot A$. If $0 \neq q \in I$ then there is a nonzero two-sided ideal U of A such that $0 \neq qU \subset A$. Thus $I \cap A \neq 0$. It follows that $\mathscr{Z}^+ \cdot A$ has finite uniform dimension. In particular, there is a bound on any finite set of orthogonal idempotents in $\mathscr{Z}^+ \cdot A$. Hence $\mathscr{Z}^+ \cdot A$ cannot contain arbitrarily large matrix rings. It follows from the theory of primitive rings [8, 1.2.1, Corollary 2] that $\mathscr{Z}^+ \cdot A \simeq M_n(D)$ where D is the finite-dimensional division algebra of Martindale's Theorem. We conclude that $\mathscr{Z}^+ \cdot A$ satisfies a standard identity. □

As one consequence, none of the algebras of differential operators covered by Corollary 3.4 (including $\mathbf{A}_1$) can be $DAlt$-symplectic. The next result suggests that some tightening of Theorem 3.10 will yield necessary and sufficient conditions.

Lemma 3.11. *Let A be a k-algebra and let K be a field extension of k. Assume that $K \otimes_k A$ is a $D\,Alt$-symplectic K-algebra such that $1 \otimes A$ is closed under the Poisson bracket. Then the bracket restricted to A arises from a $D\,Alt$-symplectic structure on A.*

Proof. We shall freely identify A and $1 \otimes A$. Suppose $\mathscr{B}$ is a basis for K/k which contains 1; for each $\lambda \in \mathscr{B}$ let $\pi_\lambda : K \to k$ be the k-linear map which projects to the coefficient of λ. The linearity of the differential symbol d implies that

$$D\,Alt^2(K \otimes_k A) \simeq K \otimes_k D\,Alt^2(A),$$

so that $\lambda \otimes a(db)(dc)$ sends $(V, W) \in Der_k(K \otimes A) \times Der_k(K \otimes A)$ to $\lambda a V(b)W(c) - \lambda a W(b)V(c)$. Each π_λ extends to a k-linear map $\hat{\pi}_\lambda : D\,Alt^2(K \otimes A) \to D\,Alt^2(A)$. Let $\omega \in D\,Alt^2(K \otimes A)$ be the alternating form which supports the symplectic structure on $K \otimes A$.

If X is a k-derivation of A it can be extended in the obvious way to $1 \otimes X$, a K-derivation of $K \otimes A$. For $X, Y \in Der_k(A)$ and $\lambda \in \mathscr{B}$, define

$$\omega_\lambda(X, Y) = \hat{\pi}_\lambda(\omega)(1 \otimes X, 1 \otimes Y).$$

If one writes out $\hat{\pi}_\lambda(\omega)$ as a differential 2-form and uses the fact that $X(A) \subseteq A$ and $Y(A) \subseteq A$, one sees that $\omega_\lambda(X, Y) \in A$. Hence,

$$\omega_\lambda(X, Y) = ((\pi_\lambda \otimes 1) \circ \omega)(1 \otimes X, 1 \otimes Y).$$

It is immediate that each $\omega_\lambda : Der_k(A) \times Der_k(A) \to A$ is k-bilinear, alternating, and satisfies the cocycle identity.

Suppose $a \in A$. For each $b \in A$,

$$ham(1 \otimes a)(1 \otimes b) = \{1 \otimes a, 1 \otimes b\} \in 1 \otimes A.$$

As a consequence, $ham(1 \otimes a)$ has the form $1 \otimes H$ for some derivation H of A. Now for each $X \in Der_k(A)$

$$\begin{aligned}\omega_\lambda(X, H) &= (\pi_\lambda \otimes 1)(\omega(1 \otimes X, 1 \otimes H)) \\ &= (\pi_\lambda \otimes 1)(\omega(1 \otimes X, ham(a))) \\ &= (\pi_\lambda \otimes 1)(1 \otimes X(a)) \\ &= \begin{cases} 0 & \text{if } \lambda \neq 1 \\ X(a) & \text{if } \lambda = 1. \end{cases}\end{aligned}$$

It follows that ω_1 defines a symplectic stucture on A with the property that $ham(1 \otimes a) = 1 \otimes ham(a)$. This equality ensures that the Poisson bracket on A induced by ω_1 corresponds to the inherited Poisson bracket on $1 \otimes A$. $\square$

Theorem 3.12. *Assume that A is a finitely generated prime k-algebra which satisfies a polynomial identity. If every derivation of A vanishes on $\mathscr{Z}(A)$, then the localization*

$\mathscr{Z}(A)^{-1}A$ can be realized as a $D\,Alt$-symplectic algebra whose Poisson bracket is the commutator.

Proof. We write $\mathscr{Z} = \mathscr{Z}(A)$. Suppose that $y_1, \ldots, y_m$ are algebra generators for A. If $X \in Der_k(\mathscr{Z}^{-1}A)$ then there are $z_j \in \mathscr{Z}$ and $u_j \in A$ such that $X(y_j) = z_j^{-1}u_j$. Set $z = z_1 z_2 \cdots z_m$. Then zX restricted to A lies in $Der_k(A)$. Hence, zX vanishes on $\mathscr{Z}$. It follows from the invertibility of z and the quotient rule that X vanishes on $\mathscr{Z}^{-1}\mathscr{Z}$.

Since k-derivations of $\mathscr{Z}^{-1}A$ vanish on the center $\mathscr{Z}^{-1}\mathscr{Z}$, we may replace A with $\mathscr{Z}^{-1}A$ and k with $\mathscr{Z}^{-1}\mathscr{Z}$. By invoking basic theorems about *p.i.* algebras we are reduced to the case that A is a finite-dimensional central simple k-algebra. That means $A = M_n(D)$ for a finite-dimensional division algebra D. Obviously, A is a Poisson algebra under the commutator bracket. Split A so that $K \otimes_k A \simeq M_n(K)$ for a field extension K. Then $K \otimes A$ is a $D\,Alt$-symplectic K-algebra compatible with the commutator, according to our differential formula for matrices over a field. Apply the lemma. $\square$

4. Commutative algebras

The basic features of a theory of symplectic structures for commutative algebras are folklore. That is to say, anyone familiar with the behavior of the algebra of C^∞-functions on a symplectic manifold would find the contents of this section entirely expected. Our exposition has considerable overlap with Loose's paper [13]; however, we take a more global point of view with a different emphasis. Our centerpiece is an observation which, at first glance, looks like an absurdly cheap way to build a symplectic form.

Theorem 4.1. *Let A be a commutative Poisson algebra. If*

$$Der_k(A) = A \cdot Ham(A),$$

then A has a compatible symplectic structure.

Proof. We would like to define $\omega : Der_k(A) \times Der_k(A) \to A$ by

$$\omega\left(\sum_i r_i ham(a_i), \sum_j s_j ham(b_j)\right) = \sum_{i,j} r_i s_j \{a_i, b_j\}.$$

To eliminate any ambiguities, we need to know that if $\sum_i r_i ham(a_i) = 0$, then $\sum_{i,j} r_i s_j \{a_i, b_j\} = 0$, for all $s_j, b_j \in A$.

This is clear because

$$\sum_{i,j} r_i s_j \{a_i, b_j\} = \sum_j s_j \sum_i r_i \{a_i, b_j\} = \sum_j s_j \left(\sum_i r_i ham(a_i)(b_j)\right).$$

Clearly, ω is an alternating A-bilinear form. We next evaluate $\omega(X, ham(b))$ for $X \in Der_k(A)$. Writing $X = \sum_i r_i ham(a_i)$,

$$\omega(X, ham(b)) = \sum_i r_i \{a_i, b\} = X(b).$$

Finally, we must show the cocycle condition. By the additivity of ω, it suffices to calculate $d\omega(X_0, X_1, X_2)$ when X_i has the form $a_i ham(b_i)$. Explicitly,

$$X_i \omega(X_j, X_k) = a_i a_j a_k \{b_i, \{b_j, b_k\}\} + a_i a_j \{b_j, b_k\}\{b_i, a_k\} + a_i a_k \{b_j, b_k\}\{b_i, a_j\},$$

$$\omega([X_i, X_j], X_k) = a_i a_j a_k \{\{b_i, b_j\}, b_k\} + a_i a_k \{b_j, b_k\}\{b_i, a_j\} - a_j a_k \{b_i, b_k\}\{b_j, a_i\}.$$

The nested brackets cancel by the Jacobi identity. A careful accounting shows that each of the other terms appears twice, once with each sign. □

It is worth observing that if $Der_k(A) = A \cdot Ham(A)$ then there is a *unique* symplectic structure compatible with the Poisson bracket.

We suspect that the condition $Der_k(A) = A \cdot Ham(A)$ is the "correct" definition of symplectic for commutative Poisson algebras. Implicitly, Loose argues that the condition is equivalent to A being Ω-symplectic. The added ingredient is the regularity of A. (This is not an unreasonable hypothesis since symplectic manifolds are smooth by definition.) We discuss some of the interrelationships among $Der_k(A) = A \cdot Ham(A)$, regularity, and symplectic structure here and at the end of Section 5.

Theorem 4.2. *If A is a regular affine symplectic algebra, then*

$$Der_k(A) = A \cdot Ham(A).$$

Proof. Since A is regular, $\Omega^1(A)$ is a finitely generated projective A-module. Then $\Omega^1(A) \simeq (\Omega^1(A))^{**} \simeq (Der_k(A))^*$ where $f \in \Omega^1(A)$ is identified with $\tilde{f} \in Alt^1(A) = (Der_k(A))^*$.

Assume that A is supported by $\omega \in Alt^2(A)$ and that $Y \in Der_k(A)$. Since $i_Y(\omega) \in Alt^1(A)$ we may write $i_Y(\omega) = \sum a_j db_j$ for some a_j and b_j in A. Recall that $-db_j = i_{ham(b_j)}(\omega)$. It follows that for

$$Z = -\sum a_j ham(b_j)$$

we have $i_Y(\omega) = i_Z(\omega)$. By nondegeneracy, $Y = Z$, i.e. $Y \in A \cdot Ham(A)$. □

Corollary 4.3. *Assume that A is a regular affine Poisson algebra. Then A has a compatible symplectic structure if and only if $Der_k(A) = A \cdot Ham(A)$.*

The corollary can be used to show quickly that if $\mathscr{G}$ is a finite-dimensional Lie algebra over k then $k[\mathscr{G}]$ is never symplectic. The underlying commutative algebra in this case is the ordinary polynomial ring $k[T_1, \ldots, T_n]$ where $T_1, \ldots, T_n$ is a basis for $\mathscr{G}$. It is well known that $M = Der_k(k[T_1, \ldots, T_n])$ is the free $K[T_1, \ldots, T_n]$-module on $\partial/\partial T_1, \ldots, \partial/\partial T_n$; as such it is a graded $k[T_1, \ldots, T_n]$-module. The tth homogeneous component, M^t, consists of all of those derivations X such that $X(T_j)$ is a homogeneous polynomial of degree t for $j = 1,\ldots,n$. For example, $ham(T_m) \in M^1$ for each m. Since $ham(ab) = a\cdot ham(b)+b\cdot ham(a)$ for all a and b in a commutative algebra, we see that

$$Ham(K[\mathcal{G}]) \subseteq \sum_{t\geq 1} M^t.$$

Hence,

$$k[\mathcal{G}]\cdot Ham(k[\mathcal{G}]) \subseteq \sum_{t\geq 1} M^t.$$

Since $k[\mathcal{G}]$ is regular, it cannot be symplectic. On the other hand, one does have a "coadjoint orbit theorem": the orbits of the appropriate associated algebraic group acting on $\mathcal{G}^*$ are all symplectic varieties ([1], and see the bibliography of [9]). If $\mathcal{G}^*$ happens to possess a dense orbit then the rational function field $k(\mathcal{G})$ is symplectic – a sheerly ring-theoretic result which can be found in [13].

We have already observed that if A is regular then symplectic is indistinguishable from Ω-symplectic. It turns out that if A is Ω-symplectic, it must be regular [13].

Theorem 4.4. *Suppose that A is a commutative Δ-symplectic algebra supported by an element of the form $\omega = \sum_{j=1}^n a_j db_j dc_j$. Then*
(i) *$A\cdot dA$ (in Δ^1) is a finitely generated projective A-module;*
(ii) *$Der_k(A)$ is a finitely generated projective A-module;*
(iii) *$Der_k(A) = A\cdot Ham(A)$.*

Proof. We argue that

$$(-a_1db_1, ham(c_1)),\ldots,(-a_ndb_n, ham(c_n)),(a_1dc_1, ham(b_n)),\ldots,(a_ndc_n, ham(b_n))$$

is a projective basis for $A\cdot dA$. (We are making the identification of i_X with X.) Suppose $h \in A$:

$$\begin{aligned}
&\sum -i_{ham(c_j)}(dh)a_jdb_j + \sum i_{ham(b_j)}(dh)a_jdc_j\\
&= \sum a_j\{h,c_j\}db_j - \sum a_j\{h,b_j\}dc_j\\
&= -i_{ham(h)}\left(\sum a_jdb_jdc_j\right) = -i_{ham(h)}(\omega) = dh.
\end{aligned}$$

(In proceeding from the second expression to the third we used the DGA-derivation formula $i_X(dbdc) = i_X(db)dc - dbi_X(dc)$.) We have proved (i).

In general, if $(f_1, ham(e_1)),\ldots,(f_m, ham(e_m))$ is a projective basis for $A\cdot dA$, then for all $t\in A$,

$$dt = \sum_p ham(e_p)(t)f_p.$$

Hence for $X \in Der_k(A)$,

$$X(t) = i_X(dt) = \sum_p ham(e_p)(t)i_X(f_p).$$

Thus,

$$X = \sum_p i_X(f_p) ham(e_p)$$

and the map sending X to $i_X(f_p)$ is A-linear. Therefore,

$$(ham(e_1), \widetilde{f}_1), \ldots, (ham(e_m), \widetilde{f}_m)$$

is a projective basis for $Der_k(A)$, where $\widetilde{f}(X) = i_X(f)$. Properties (ii) and (iii) follow. □

Corollary 4.5 (Loose [13]). *Assume that A is an affine commutative algebra. If A is Ω-symplectic, then A is regular.*

Proof. In this case $A \cdot dA = \Omega^1(A)$. Apply (i) of the theorem. □

The corollary may be true when A is simply symplectic. The Lipman–Zariski conjecture states that (for reduced affine algebras) if $Der_k(A)$ is a finitely generated projective A-module, then A is regular.

The classical example of a symplectic manifold is the cotangent bundle. The analogous algebraic construction for a commutative regular domain is also symplectic; a proof can be found in [13]. We will present another argument which avoids localization and explicitly constructs derivations.

Let A be a (commutative) regular k-affine domain. Form the ring of differential operators $\mathscr{D}(A) = \mathbf{D}(A)$, a naturally filtered algebra. Its associated graded algebra is the affine algebra $A[Der_k(A))]$ where $A[M]$ denotes the symmetric A-algebra on the module M [14, 15.4.5]. Hence, the commutator on $\mathscr{D}(A)$ makes $A[Der_k(A)]$ into a Poisson algebra; the bracket is the unique one such that

$$\{a, b\} = 0 \quad \text{for } a, b \in A,$$

$$\{Y, b\} = Y(b) \quad \text{for } Y \in Der_k(A) \text{ and } b \in A,$$

$$\{Y, Z\} = Y \circ Z - Z \circ Y \quad \text{for } Y, Z \in Der_k(A).$$

(We warn the reader that there are two associative products for elements of $Der_k(A)$: composition and the commutative product in $A[Der_k(A)]$. For this reason we will take care to use the small circle for composition and we will write $\{*, *\}$ for the Lie bracket on $Der_k(A)$.) The first step in analyzing the algebra $A[Der_k(A)]$ is to construct derivations on a symmetric algebra. The argument is routine.

Lemma 4.6. *Let A be a commutative k-algebra and let M be an A-module. Assume $E_0 \in Der_k(A)$ and $E_1 : M \to A \oplus M$ is a k-linear map such that $E_1(am) = E_0(a)m + aE_1(m)$. Then there exists $E \in Der_k(A[M])$ with $E|A = E_0$ and $E|M = E_1$.*

From now on we will assume A is a regular k-affine domain and set B to be the "cotangent algebra" $gr\mathscr{D}(A) = A[Der_k(A)]$. We repeatedly exploit the observation that a differential or derivation on B is determined by its behavior on $A \cup Der_k(A)$. By regularity, $\Omega^1(A)$ is a finitely generated projective A-module, say with projective basis $(f_1,X_1),\ldots,(f_n,X_n)$. Moreover, $(X_1,\tilde{f}_1),\ldots,(X_n,\tilde{f}_n)$ is a projective basis for $Der_k(A)$.

Use the lemma to define $\partial_t \in Der_k(B)$ for $t = 1,\ldots,n$ so that $\partial_t(a) = 0$ for $a \in A$ and $\partial_t(Y) = \tilde{f}_t(Y)$ for $Y \in Der_k(A)$. Apply the lemma a second time: for each $Z \in Der_k(A)$, define an extension to $\hat{Z}$ in $Der_k(B)$ by $\hat{Z}(Y) = \sum_t Z(\tilde{f}_t(Y))X_t$ for each $Y \in Der_k(A)$.

Proposition 4.7. *If $\Omega^1(A)$ has a projective basis $(f_1,X_1),\ldots,(f_n,X_n)$, then $\Omega^1(B)$ is a finitely generated projective B-module with basis*

$$(dX_1,\partial_1),\ldots,(dX_n,\partial_n),(f_1,\hat{X}_1),\ldots,(f_n,\hat{X}_n).$$

In particular, B is regular.

Proof. It suffices to show that

$$\sum_j \partial_j(v)dX_j + \sum_j \hat{X}_j(v)f_j = dv$$

for generators v of B in $A \cup Der_k(A)$.

Suppose $v = a$ lies in A. By construction, $\partial_t(a) = 0$. Therefore,

$$\sum_j \partial_j(a)dX_j + \sum_j \hat{X}_j(a)f_j = \sum_j X_j(a)f_j = da.$$

Suppose $v = Y$ lies in $Der_k(A)$. By construction, $\partial_t(Y) = \tilde{f}_t(Y)$. Using the formula for $\hat{X}_j$ and the fact that X_t and $X_j(\tilde{f}_t(Y))$ commute in the algebra B, we obtain

$$\begin{aligned}
\sum_j \partial_j(Y)dX_j + \sum_j \hat{X}_j(Y)f_j &= \sum_j \tilde{f}_j(Y)dX_j + \sum_j\sum_t X_j(\tilde{f}_t(Y))X_t f_j \\
&= \sum_j \tilde{f}_j(Y)dX_j + \sum_t X_t \sum_j X_j(\tilde{f}_t(Y))f_j \\
&= \sum_j \tilde{f}_j(Y)dX_j + \sum_t X_t d(\tilde{f}_t(Y)) \\
&= \sum_j d(\tilde{f}_j(Y)X_j) - \sum_j X_j d(\tilde{f}_j(Y)) + \sum_t X_t d(\tilde{f}_t(Y)) \\
&= d\left(\sum_j \tilde{f}_j(Y)X_j\right).
\end{aligned}$$

This last expression is dY. □

Corollary 4.3 tells us that we will know that B is symplectic once we establish the next assertion.

Proposition 4.8. *$Der_k(B) = B \cdot Ham(B)$.*

Proof. According to the previous proposition, $Der_k(B)$ is generated as a B-module by $\partial_1, \dots, \partial_n, \hat{X}_1, \dots, \hat{X}_n$. So it suffices to show that each ∂_t and $\hat{X}_t$ lie in $B \cdot Ham(B)$. We test derivations of B on $A \cup Der_k(A)$.

If $f \in \Omega^1(A)$ is written $f = \sum a_s db_s$ then it induces a well-defined derivation on B,

$$f^\sharp = \sum a_s ham(b_s).$$

Then $f^\sharp(c) = 0$ for all $c \in A$. If $Y \in Der_k(A)$ then

$$f^\sharp(Y) = \sum a_s\{b_s, Y\} = -\sum a_s Y(b_s) = -\tilde{f}(Y).$$

For the special case $f = f_t$ we conclude that $\partial_t = (-f_t)^\sharp \in B \cdot Ham(B)$.

Next we look at any $Z \in Der_k(A)$ and compute

$$ham(Z) - \sum_t \{Z, X_t\}\partial_t.$$

When restricted to $a \in A$, the evaluation of the expression is $\{Z, a\}$. This is the same as $Z(a)$ or, equivalently, $\hat{Z}(a)$. When we evaluate the expression on Y in $Der_k(A)$ we get

$$\begin{aligned} &Z \circ Y - Y \circ Z - \sum_t \tilde{f}_t(Y) Z \circ X_t + \sum_t \tilde{f}_t(Y) X_t \circ Z \\ &= Z \circ Y - Y \circ Z - \sum_t \tilde{f}_t(Y) Z \circ X_t + Y \circ Z \\ &= Z \circ Y - Z \circ \left(\sum_t \tilde{f}_t(Y) X_t \right) + \sum_t Z(\tilde{f}_t(Y)) X_t = Z \circ Y - Z \circ Y + \hat{Z}(Y). \end{aligned}$$

We conclude that

$$ham(Z) - \sum_t \{Z, X_t\}\partial_t = \hat{Z}.$$

It follows that $\hat{Z} \in B \cdot Ham(B)$. $\square$

If $\mathfrak{B}$ is an alternating bilinear form on the finite-dimensional vector space V, we can now determine when $k[\mathfrak{B}]$ is symplectic. We claim it is symplectic if and only if $\mathfrak{B}$ is nondegenerate. Suppose that there is a nonzero T in V such that $\mathfrak{B}(T, v) = 0$ for all $v \in V$. Then $T \in \mathscr{PZ}(k[\mathfrak{B}])$. But $(\partial/\partial T)(T) = 1$. Since there is an element of the Poisson center which does not vanish under all derivations, $k[\mathfrak{B}]$ is not symplectic (Proposition 2.6). Conversely, if $\mathfrak{B}$ is nondegenerate then we can find a basis for V so that $k[\mathfrak{B}]$ is the polynomial algebra in $2n$ variables $k[T_1, \dots, T_n, S_1, \dots, S_n]$ where

$$\mathfrak{B}(T_i, T_j) = \mathfrak{B}(S_i, S_j) = \mathfrak{B}(S_i, T_j) = 0 \quad \text{for all } i \neq j; \quad \text{and} \quad \mathfrak{B}(S_i, T_i) = 1.$$

Clearly, $k[\mathfrak{B}]$ is isomorphic as a k-algebra (ignore the grading!) to the symplectic algebra $gr\mathscr{D}(k[T_1, \dots, T_n])$.

We close this section by analyzing the "singular" cotangent algebra for the cusp. The associated graded ring in question is a Poisson subalgebra of the symplectic plane $k[T,S]$, namely

$$B = k + T^2k[T,S] + S^2k[T,S] + TSk[T,S].$$

(a) Every derivation of B extends to a derivation of $k[T,S]$. If $\zeta \in Der_k(B)$ then ζ certainly extends to $k(T,S)$. A priori, $\zeta(T)$ may lie in $k(T,S)$. However,

$$T\zeta(T) = \tfrac{1}{2}\zeta(T^2) \in B$$

and

$$S^2\zeta(T) = S\zeta(TS) - \tfrac{1}{2}T\zeta(S^2) \in B.$$

Thus, $S^2(T\zeta(T)) = T(S^2\zeta(T))$ exhibits two factorizations in $k[T,S]$. Since the polynomial ring is a *UFD*, we have $T|T\zeta(T)$ in $k[T,S]$. It follows that $\zeta(T) \in k[T,S]$. Similarly, $\zeta(S) \in k[T,S]$.

(b) Every derivation of B can be written uniquely in the form $a(\partial/\partial T) + b(\partial/\partial S)$ where $a, b \in k[T,S]$ and neither a nor b has a constant term. Since $k[T,S]$ is a polynomial algebra, its derivations comprise a free module of rank 2 on $(\partial/\partial T)$ and $(\partial/\partial S)$. If $a(\partial/\partial T) + b(\partial/\partial S)$ is a derivation which stabilizes B then

$$a\frac{\partial}{\partial T}(T^2) + b\frac{\partial}{\partial S}(T^2) \in B.$$

That is,

$$aT \in B.$$

Thus, a has no constant term. The same argument can be made for b. Conversely, all derivations of the required form do stabilize B. As to uniqueness, if the restriction of $a(\partial/\partial T) + b(\partial/\partial S)$ to B is zero then $aT = 0$ and $bS = 0$.

(c) B is a symplectic algebra compatible with the bracket. The idea is to restrict the symplectic form on $k[T,S]$ to B. Recall that $ham(T) = -(\partial/\partial S)$ and $ham(S) = (\partial/\partial T)$. Thus, the symplectic structure on $k[T,S]$ is supported by

$$\begin{aligned}\omega\left(a\frac{\partial}{\partial T} + b\frac{\partial}{\partial S}, c\frac{\partial}{\partial T} + d\frac{\partial}{\partial S}\right) &= \omega(a\, ham(S) - b\ ham(T), c\ ham(S) - d\ ham(T)) \\ &= bc - ad.\end{aligned}$$

If the two derivations above lie in $Der_k(B)$ then the total degree of bc, as well as ad, is at least 2. In other words, ω takes its values in B!

(d) $Der_k(B) \neq B \cdot Ham(B)$. Obviously, $T\frac{\partial}{\partial T}$ is a derivation of B. By degree considerations, $T\frac{\partial}{\partial T}$ must lie in $Ham(B)$ if it has any chance of belonging to $B \cdot Ham(B)$. On the other hand, the computation

$$ham(T^iS^j) = jT^iS^{j-1}\frac{\partial}{\partial T} - iT^{i-1}S^j\frac{\partial}{\partial S}$$

shows that any expression in $Ham(B)$ which has $T(\partial/\partial T)$ in its "support" must also have $S(\partial/\partial S)$.

We reiterate. There are commutative affine domains B which are symplectic but which fail to satisfy $Der_k(B) = B \cdot Ham(B)$. Based on analogy with Corollary 3.4, we always expect $gr\mathscr{D}(A)$ to be symplectic (in the Alt sense) for an affine domain A. In contrast, it is possible to modify the argument which will appear in Proposition 5.3 to prove that if Der is generated by Ham as a $gr\mathscr{D}(A)$-module, then $Der_k(A)$ is a finitely generated projective A-module.

5. Symplectic potentials

Suppose that A is a Δ-symplectic k-algebra supported by ω. One way to guarantee that $d\omega = 0$ is to find an exact ω. That is, one may have a $\theta \in \Delta^1$ with $d\theta = \omega$. In this situation θ is called a *Δ-symplectic potential.* According to Theorem 2.2b, a Δ-symplectic potential always induces a symplectic potential. If $\widetilde{\omega} = d\widetilde{\theta}$ then for $a, b \in A$,

$$\begin{aligned}\{a,b\} &= \widetilde{\omega}(ham(a), ham(b))\\ &= ham(a)\widetilde{\theta}(ham(b)) - ham(b)\widetilde{\theta}(ham(a)) - \widetilde{\theta}([ham(a), ham(b)])\\ &= \{a, \widetilde{\theta} \circ ham(b)\} + \{\widetilde{\theta} \circ ham(a), b\} - \widetilde{\theta} \circ ham(\{a,b\}).\end{aligned}$$

Thus there is a k-linear map $f = \widetilde{\theta} \circ ham : A \to A$ such that

$$\{a,b\} = \{a, f(b)\} + \{f(a), b\} - f(\{a,b\}). \qquad (\dagger)$$

Notice that f vanishes on $\mathscr{P}\mathscr{Z}(A)$ because ham does.

Now assume that A is commutative. Then $\widetilde{\theta}$ is an A-module homomorphism; in this case f lies in $Der_k(A)$. (Huebschmann [9] says A has a "Poisson potential" when ($\dagger$) holds for some $f \in Der_k(A)$. We do not know of a good definition for Poisson potential when A is not commutative.)

Theorem 5.1. (a) *If the symplectic algebra A is supported by a potential, then there exists a map $T : A \to A$ which is a derivation of the Poisson-bracket Lie algebra and which satisfies $T|\mathscr{P}\mathscr{Z}(A) = id$.*

(b) *Assume A is an algebra all of whose derivations are inner. If there is a Lie derivation of A under the commutator bracket which is the identity on $\mathscr{Z}(A)$, then the commutator symplectic structure is supported by a potential.*

Proof. (a) We assume that the symplectic structure is supported by $d\theta$ with $\theta \in Alt^1(A)$. Set $f = \theta \circ ham$, so that equation ($\dagger$) holds. Define T by $T(c) = c - f(c)$ for all $c \in A$. If one replaces $f(a)$ with $a - T(a)$ and $f(b)$ with $b - T(b)$ in ($\dagger$), one obtains

$$T(\{a,b\}) = \{T(a), b\} + \{a, T(b)\}.$$

Since f vanishes on $\mathscr{P}\mathscr{Z}(A)$, we see that $T(c) = c$ for all $c \in \mathscr{P}\mathscr{Z}(A)$.

(b) If T is the requisite derivation then we may reverse the argument of part a to construct a k-endomorphism $f : A \to A$ which satisifies (†) for the commutator bracket and vanishes on $\mathscr{Z}(A)$. Since ad induces an isomorphism $A/\mathscr{Z}(A) \to Der_k(A)$ and $f(\mathscr{Z}(A)) = 0$, there is an element $\phi \in Alt^1(A)$ with $\phi \circ ad = f$. It follows from the derivation of (†) that $d\phi$ supports the commutator symplectic structure. □

The connection is known to be much tighter when $A = C^\infty(M)$. Avez et al. (cf. [12, p. 33]) prove that if M is noncompact and θ is a symplectic potential, then every Lie derivation of A has the form $X + \lambda(id - \dot{\theta})$ where X is a Poisson derivation (and so is an associative algebra derivation), λ is a scalar, and $\dot{\theta}$ pulls back θ. If ω does not support a potential, they conclude that every Lie derivation of A lies in $Der_k(A)$. Better yet, even if M may be compact, it is generally true that every Lie derivation of A which vanishes on 1 is an associative derivation.

If one examines not-commutative algebras, then these results about Lie derivations are reminiscent of conjectures made by Herstein in [7]. For example, Kaplansky and Martindale [15] prove that every Lie derivation of $M_n(k)$ takes the form $X + \lambda \cdot tr$ where X is an associative derivation, $\lambda \in k$, and tr denotes the trace. Notice that the symplectic form given in section 3 is $\omega = d\theta$ where

$$\theta = -\frac{1}{n}\sum_{i,j} E_{ij}(dE_{ji}).$$

A calculation shows that the function f in part (a) takes the form

$$-\frac{1}{n}\sum_{i,j} E_{ij}(dE_{ji})(ad(b)) = b - \frac{1}{n}tr(b) \cdot I$$

for all $b \in M_n(k)$.

Example 1. If $\mathscr{G}$ is a finite-dimensional semisimple Lie algebra, then the commutator Poisson bracket on $U(\mathscr{G})$ is compatible with a symplectic structure supported by a potential.

We will need standard facts about the enveloping algebra, which can be found in [5]. Write $U = U(\mathscr{G})$. First, every associative algebra derivation of U is inner. (If $D \in Der_k(U)$, then $D(\mathscr{G}) \subseteq U_n$ for some n, where U_n is the finite-dimensional $\mathscr{G}$-module spanned by products of m elements from $\mathscr{G}$ with $m \le n$. Thus $D \in H^1(\mathscr{G}, U_n)$. But the cohomology group vanishes by Whitehead's Lemma and semisimplicity; there is an $a \in U_n$ with $D(x) = [a,x]$ for all $x \in \mathscr{G}$. The equality extends to all $x \in U$.) Second,

$$U = \mathscr{Z}(U) \oplus [U,U]$$

as vector spaces. Define T to be the vector space projection of U onto $\mathscr{Z}(U)$. Then

$$T([a,b]) = 0 = [T(a),b] + [a,T(b)] \quad \text{for all } a,b \in U$$

and $T|\mathscr{Z}(U) = id$. Apply Theorem 5.1(b).

Example 2. $\mathbf{A}_1$ has no symplectic structure supported by a potential.

Dixmier [5] proves that all derivations of $\mathbf{A}_1$ are inner. In [6], the author conjectures that the commutator symplectic structure on $\mathbf{A}_1$ fails to have a $LD\,Alt$-potential. As we saw in Section 4, the conjecture amounts to proving that there is no potential (i.e., not even one in Alt) for the commutator structure. By Section 1, once we have the assertion for the Poisson bracket coinciding with the commutator bracket, we have the general claim made in the statement of the example. But Joseph [10] has proved that all Lie derivations of $\mathbf{A}_1$ are inner. They must vanish on 1. Thus, there is no potential by Theorem 5.1(a).

For Example 3, we will show the known result (see [13]) that the natural symplectic structure on the "cotangent algebra" $gr\mathscr{D}(A)$ (for a commutative regular affine domain A) is supported by a potential. It turns out that this is true for transparent algebraic reasons.

Definition. B is a *graded Poisson algebra of degree* $-s$ if it is an $\mathbb{N}$-graded algebra with a Poisson bracket such that $\{B_m, B_n\} \subset B_{m+n-s}$ for all $m, n \geq 0$. (Here we set $B_r = 0$ whenever $r < 0$.)

The algebra $gr\mathscr{D}(A)$ is a graded Poisson algebra of degree -1 (even when A is not regular).

Theorem 5.2. *Let B be a commutative graded Poisson algebra of degree -1 such that $Der_k(B) = B \cdot Ham(B)$. Then the unique symplectic structure on B compatible with the Poisson bracket is supported by a potential.*

Proof. The potential comes from the Euler derivation, which exists on any graded algebra. If R is a graded algebra define $D : R \to R$ on nonzero homogeneous elements by $D(r) = (deg\ r)r$. It is obvious that D is a derivation.

Since $Der_k(B) = B \cdot Ham(B)$ there is a unique $\omega \in Alt^2(B)$ which is compatible with the Poisson bracket (see Section 4). We claim that $di_D(\omega) = \omega$. To check this, we need only evaluate both sides on $Ham(B) \times Ham(B)$,

$$\begin{aligned} di_D\omega(ham(a), ham(b)) &= ham(a)(i_D\omega(ham(b)) - ham(b)(i_D\omega(ham(a))) \\ &\quad - i_D\omega([ham(a), ham(b)]) \\ &= \{a, \omega(D, ham(b))\} - \{b, \omega(D, ham(a))\} \\ &\quad - \omega(D, ham\{a, b\}) \\ &= \{a, D(b)\} - \{b, D(a)\} - D(\{a, b\}). \end{aligned}$$

There is no loss of generality in assuming that a and b are homogeneous. Thus,

$$\begin{aligned} di_D\omega(ham(a),ham(b)) &= (degb)\{a,b\} + (dega)\{a,b\} - (dega + degb - 1)\{a,b\} \\ &= \{a,b\} \\ &= \omega(ham(a),ham(b)). \quad \square \end{aligned}$$

Example 3. If A is a commutative regular affine domain, then the natural symplectic structure on $gr\mathscr{D}(A)$ is supported by a potential.

Set $B = gr\mathscr{D}(A)$. Then B is regular and so, by Theorem 4.2, $Der_k(B) = B \cdot Ham(B)$. The assertion follows from the theorem above. There is a more traditional differential formula for the symplectic potential. Let $(f_1,X_1),\ldots,(f_n,X_n)$ be a projective basis for $\Omega^1(A)$, where $X_j \in Der_k(A)$. We claim that $\sum_{j=1}^n X_j f_j \in \Omega^1(B)$ "coincides" with $i_D(\omega)$ under the identification of $\Omega^1(B)$ and $Alt^1(B)$ arising from regularity. We must show that

$$i_Y\left(\sum X_j f_j\right) = \omega(D,Y) \quad \text{for all } Y \in Der_k(B).$$

Once again, we need only test equality for $Y \in Ham(B)$. We can reduce further by only evaluating at $Y = ham(b)$ for b from an algebra generating set for B. For example, we need only consider $b \in A \cup Der_k(A)$. If $a \in A$ then $i_{ham(a)}(f_j) = 0$ because $\{A,A\} = 0$; notice that $\omega(D,ham(a)) = D(a) = 0$ since $a \in B^0$. Thus,

$$i_{ham(a)}\left(\sum X_j f_j\right) = 0 = \omega(D,ham(a)).$$

If $Z \in Der_k(A)$ then

$$Z = \sum \tilde{f}_j(Z)X_j$$

because $(X_1,\tilde{f}_j),\ldots,(X_n,\tilde{f}_n)$ is a projective basis for $Der_k(A)$. That is,

$$Z = i_Z\left(\sum X_j f_j\right).$$

On the other hand, $\omega(D,ham(Z)) = D(Z) = Z$. Thus,

$$i_Z\left(\sum X_j f_j\right) = Z = \omega(D,ham(Z)).$$

As we discussed in the paragraph following Theorem 5.1, the existence of a potential should tell us something about the structure of Lie derivations. In the very special case when A is the polynomial ring in one variable, $gr\mathscr{D}(A)$ is the symplectic plane. It is known [10] in this case that every Lie derivation looks like $\zeta + \lambda(id - D)$ where ζ lies in Ham, λ is a scalar, and D is the Euler derivation.

From the algebraic point of view, it is not obvious what is gained by having a symplectic potential. We illustrate the power of this added information in the next result.

Proposition 5.3. *Let A be a commutative Poisson algebra and assume that $Der_k(A) = A \cdot Ham(A)$. If the unique compatible symplectic structure is supported by a potential, then $Der_k(A)$ is a finitely generated projective A-module.*

Proof. Let $\theta \in Alt^1(A)$ be the symplectic potential and define $E \in Der_k(A)$ by $E = -\theta \circ ham$. Since every derivation lies in the module generated by $Ham(A)$, we may write

$$E = \sum r_j ham(s_j) \quad \text{for some } r_j, s_j \in A.$$

We argue that $\theta = \sum r_j ds_j$ in $Alt^1(A)$. As usual, we need only test the equality by evaluation on derivations in $Ham(A)$. If $a \in A$,

$$\begin{aligned}\left(\sum r_j ds_j\right)(ham(a)) &= \sum r_j \{a, s_j\} \\ &= -\sum r_j ham(s_j)(a) \\ &= -E(a) \\ &= \theta(ham(a)).\end{aligned}$$

Consequently, the symplectic structure on A is supported by $d\theta \in D\,Alt(A)$. The conclusion follows from Theorem 4.4. □

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